18.05 Exam 1 Solutions

Problem 0. (5 pts)
Turn in notecard.

Problem 1. (20 pts: 4,4,4,8)
(a) (Produced by counting reboots)

		R				
		1	2	3	8	
C	Mac = 0	1/6	2/6	0	1/6	4/6
	PC = 1	0	1/6	1/6	0	2/6
		1/6	3/6	1/6	1/6	1

(b) From the table: $E(C) = 0 \cdot \frac{4}{6} + 1 \cdot \frac{2}{6} = \boxed{\frac{1}{3}.}$ $\quad E(R) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{3}{6} + 3 \cdot \frac{1}{6} + 8 \cdot \frac{1}{6} = \frac{18}{6} = \boxed{3.}$

(c) We use the formula $\text{Cov}(C, R) = E(CR) - E(C)E(R)$.

$E(CR) = 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} = \frac{5}{6} \;\Rightarrow\; \text{Cov}(C, R) = \frac{5}{6} - 1 = \boxed{-\frac{1}{6}.}$

Since covariance is not zero, they are not independent.

The negative covariance suggests that as C increases R tends to decrease. That is, PC users have to reboot less often than Mac users.

(d) (i) Independendent $\Rightarrow$ joint pmf = product of marginal pmf's.

(ii) $P(W > M) =$ sum of red prob. in table $= \frac{1}{8} + \frac{1}{8} + \frac{1}{4} = \boxed{\frac{1}{2}.}$

(iii) $\text{Cor}(W, M) = 0$ since they are independent.

		W		
		2	3	
	1	**1/8**	**1/8**	1/4
M	2	1/4	**1/4**	1/2
	8	1/8	1/8	1/4
		1/2	1/2	1

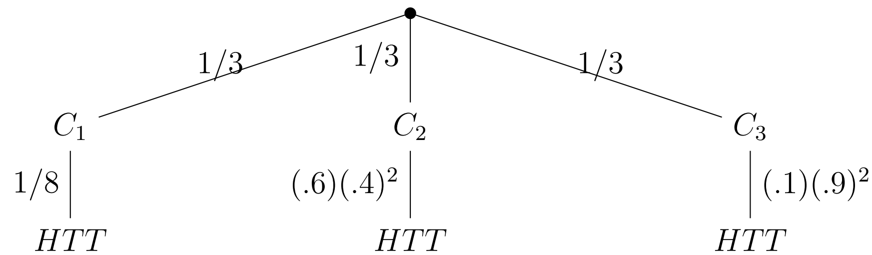
Problem 2. (8 pts)
We are given that $T = \frac{5}{9}X - \frac{160}{9}$, and $S = \frac{5}{9}Y - \frac{160}{9}$.

The algebraic properties of covariance say that $\text{Cov}(aX + b, cY + d) = ac\text{Cov}(X, Y)$.
Thus,

$$\text{Cov}(T, S) = \frac{5}{9} \cdot \frac{5}{9}\text{Cov}(X, Y) = \left(\frac{5}{9}\right)^2 \cdot 4 = \boxed{\frac{100}{81}.}$$

$\boxed{\rho(T, S) = \rho(X, Y) = 0.8}$, since correlation is scale and shift invariant.

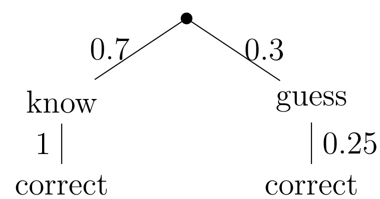
Problem 3. (16 pts: 8,8)
(a) Let C_1, C_2, C_3 be the .5, .6, .1 coins respectively. We use a tree to represent the law of total probability. (We only include the paths on the tree we are interested in.)



So, $P(HTT) = \frac{1}{3}\left(\frac{1}{8} + (.6)(.4)^2 + (.1)(.9)^2\right) = \frac{1}{3}\left(\frac{125}{1000} + \frac{96}{1000} + \frac{81}{1000}\right) = \boxed{\frac{302}{3000}}.$

(b) Bayes' Rule: $P(C_1|HTT) = \frac{P(HTT|C_1)\cdot P(C_1)}{P(HTT)} = \boxed{\frac{\frac{1}{8}\cdot\frac{1}{3}}{P(HTT)}} = \frac{3000}{3\cdot 8\cdot 302} = \frac{125}{302}.$

Problem 4. (16 pts: 4,4,4,4)



(a) P(random question is correct) = 0.7 + (0.25)(0.3) = 0.775.
Let X_j be success on question j, so $X_j \sim \text{Bernoulli}(0.775)$.
Let $\overline{X}$ = average of the X_j = score on exam.
$E(\overline{X}) = E(X_j) = 0.775.$ $\boxed{\textbf{Answer:}\ 77.5\%}.$

(b) Let Y = number correct $\sim \text{Binom}(10, p)$.

$$\boxed{P(Y \geq 9) = \binom{10}{9} p^9(1-p) + p^{10} = \binom{10}{9}(0.775)^9(0.225) + (0.775)^{10}.}$$

(c) $\boxed{\textbf{Answer:}\ p^6 = (0.775)^6}.$

(d) I'd rather have a test with 10 questions, since the more questions the more likely I'll score close to the mean (law of large numbers), which at 77.5% is too low.

Problem 5. (20 pts: 4,4,4,4,4)

(a) Need $\int_0^3 f_X(x)\,dx = 1 \Rightarrow \int_0^3 kx^2\,dx = 1 \Rightarrow \frac{k3^3}{3} = 1 \Rightarrow \boxed{k = \frac{1}{9}}.$

For $0 \leq x \leq 3$, $F_X(x) = \int_0^x ku^2\,du = \frac{kx^3}{3} = \boxed{\frac{x^3}{27}}.$

Outside of [0,3]: $F_X(x) = 0$ for $x < 0$ and $F_X(x) = 1$ for $x > 2$.

(b) $F_X(q_{0.3}) = 0.3 \Rightarrow \frac{q_{0.3}^3}{27} = 0.3 \Rightarrow \boxed{q_{0.3} = (8.1)^{1/3}}.$

(c) $E(Y) = E(X^3) = \int_0^3 x^3 f_X(x)\,dx = \frac{1}{9}\int_0^3 x^5\,dx = \frac{3^6}{54} = \boxed{\frac{27}{2}}.$

(d) $\text{Var}(Y) = E((Y-\mu_Y)^2) = \boxed{\int_0^3 \left(x^3 - \frac{27}{2}\right)^2 \frac{x^2}{9}\,dx}$

Or, $\text{Var}(Y) = E(Y^2) - \left(\frac{27}{2}\right)^2 = E(X^6) - \frac{27^2}{4} = \boxed{\int_0^3 x^6 \cdot \frac{x^2}{9}\,dx - \frac{27^2}{4}} = 3^5 - 3^6/4 = 243/4.$

(e) $F_Y(y) = P(Y \le y) = P(X \le y^{1/3}) = F_X(y^{1/3}) = \dfrac{y}{27} \Rightarrow \boxed{f_Y(y) = \dfrac{d}{dy}F_y = \dfrac{1}{27}, \text{ on } [0, 27]}.$

Alternatively, $f_Y(y) = \dfrac{d}{dy}F_Y(y) = F_X'(y^{1/3})\,\dfrac{1}{3}y^{-2/3} = f_X(y^{1/3})\,\dfrac{1}{3}y^{-2/3} = \dfrac{1}{9}y^{2/3}\dfrac{1}{3}y^{-2/3} = \dfrac{1}{27}.$

Problem 6. (15 pts: 10,5)

(a) Let S be the total number of minutes they are late for the year. The problem asks for $P(S > 630)$.

Let X_i = how late they are on the ith day: $X_i \sim \exp(1/6)$. We know $E(X_i) = 6$, $\mathrm{Var}(X_i) = 6^2$.

We have $S = \sum_{i=1}^{100} X_i$ and since (we assume) the X_i are i.i.d. we have

$$E(S) = 600, \quad \mathrm{Var}(S) = 6^2 100, \quad \sigma_S = 60.$$

The central limit theorem says that standardized S is approximately standard normal. So,

$$P(S > 630) = P\left(\frac{S - 600}{60} > \frac{630 - 600}{60}\right) \approx P(Z > 1/2) = 1 - \Phi(0.5) = 0.309.$$

The last value was found by using the table of standard normal probabilities

(b) Let T be the number of minutes they are late on a random day. The problem asks for

$$E(T^2 + T) = \int_0^\infty (t^2 + t)\frac{1}{6}\mathrm{e}^{-t/6}\,dt.$$