# Joint Distributions, Independence Covariance and Correlation

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| $X\backslash Y$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 |
| 2 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 |
| 3 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 |
| 4 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 |
| 5 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 |
| 6 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 |

# Central Limit Theorem

**Setting:** $X_1, X_2, \ldots$ i.i.d. with mean $\mu$ and standard dev. $\sigma$.

For each $n$:

$$\overline{X}_n = \frac{1}{n}(X_1 + X_2 + \ldots + X_n)$$
$$S_n = X_1 + X_2 + \ldots + X_n.$$

**Conclusion:** For large $n$:

$$\overline{X}_n \approx \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$$
$$S_n \approx \mathrm{N}\left(n\mu, n\sigma^2\right)$$

Standardized $S_n$ or $\overline{X}_n \approx \mathrm{N}(0, 1)$

# Board Question: CLT

1. Carefully write the statement of the central limit theorem.

2. To head the newly formed US Dept. of Statistics, suppose that 50% of the population supports Erika, 25% supports Ruthi, and the remaining 25% is split evenly between Peter, Jon and Jerry.

A poll asks 400 random people who they support. What is the probability that at least 55% of those polled prefer Erika?

3. (Not for class. Solution will be on posted slides.)
An accountant rounds to the nearest dollar. We'll assume the error in rounding is uniform on [-.5, .5]. Estimate the probability that the total error in 300 entries is more than $5.

# Joint Distributions

$X$ and $Y$ are *jointly distributed* random variables.

Discrete: Probability mass function (pmf):

$$p(x_i, y_j)$$

Continuous: probability density function (pdf):

$$f(x, y)$$

Both: cumulative distribution function (cdf):

$$F(x, y) = P(X \leq x, Y \leq y)$$

# Discrete joint pmf: example 1

Roll two dice: $X = \#$ on first die, $Y = \#$ on second die

$X$ takes values in 1, 2, ..., 6, $Y$ takes values in 1, 2, ..., 6

**Joint probability table:**

| $X \backslash Y$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 |
| 2 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 |
| 3 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 |
| 4 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 |
| 5 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 |
| 6 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 |

pmf: $p(i, j) = 1/36$ for any $i$ and $j$ between 1 and 6.

# Discrete joint pmf: example 2

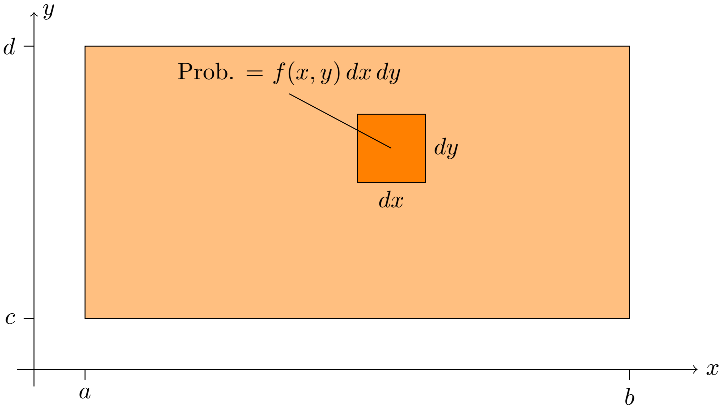
Roll two dice: $X = \#$ on first die, $T =$ total on both dice

| $X\backslash T$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 0 | 0 | 0 | 0 | 0 |
| 2 | 0 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 0 | 0 | 0 | 0 |
| 3 | 0 | 0 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 0 | 0 | 0 |
| 4 | 0 | 0 | 0 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 0 | 0 |
| 5 | 0 | 0 | 0 | 0 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 0 |
| 6 | 0 | 0 | 0 | 0 | 0 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 |

# Continuous joint distributions

- $X$ takes values in $[a, b]$, $Y$ takes values in $[c, d]$
- $(X, Y)$ takes values in $[a, b] \times [c, d]$.
- *Joint probability* **density** *function* (pdf) $f(x, y)$

$f(x, y)\,dx\,dy$ is the probability of being in the small square.



# Properties of the joint pmf and pdf

**Discrete case: probability mass function (pmf)**

1. $0 \le p(x_i, y_j) \le 1$
2. Total probability is 1.

$$\sum_{i=1}^{n} \sum_{j=1}^{m} p(x_i, y_j) = 1$$

**Continuous case: probability density function (pdf)**

1. $0 \le f(x, y)$
2. Total probability is 1.

$$\int_c^d \int_a^b f(x, y)\, dx\, dy = 1$$

Note: $f(x, y)$ can be greater than 1: it is a density *not* a probability.

# Example: discrete events

Roll two dice: $X = \#$ on first die, $Y = \#$ on second die.

Consider the event: $A =$ '$Y - X \geq 2$'

Describe the event $A$ and find its probability.

# Example: discrete events

Roll two dice: $X = \#$ on first die, $Y = \#$ on second die.

Consider the event: $A =$ '$Y - X \geq 2$'

Describe the event $A$ and find its probability.

**answer:** We can describe $A$ as a set of $(X, Y)$ pairs:

$$A = \{(1,3),(1,4),(1,5),(1,6),(2,4),(2,5),(2,6),(3,5),(3,6),(4,6)\}.$$

Or we can visualize it by shading the table:

| $X \backslash Y$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 |
| 2 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 |
| 3 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 |
| 4 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 |
| 5 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 |
| 6 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 |

$P(A) =$ sum of probabilities in shaded cells $= 10/36$.

# Example: continuous events

Suppose $(X, Y)$ takes values in $[0, 1] \times [0, 1]$.

Uniform density $f(x, y) = 1$.

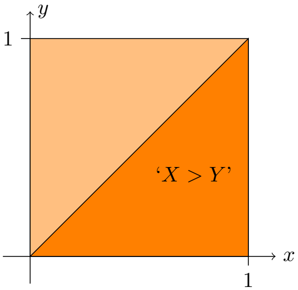
Visualize the event '$X > Y$' and find its probability.

# Example: continuous events

Suppose $(X, Y)$ takes values in $[0,1] \times [0,1]$.

Uniform density $f(x, y) = 1$.

Visualize the event '$X > Y$' and find its probability.
**answer:**




The event takes up half the square. Since the density is uniform this is half the probability. That is, $P(X > Y) = .5$

# Cumulative distribution function

$$F(x, y) = P(X \leq x,\ Y \leq y) = \int_c^y \int_a^x f(u, v)\, du\, dv.$$

$$f(x, y) = \frac{\partial^2 F}{\partial x \partial y}(x, y).$$

**Properties**

1. $F(x, y)$ is non-decreasing. That is, as $x$ or $y$ increases $F(x, y)$ increases or remains constant.
2. $F(x, y) = 0$ at the lower left of its range.
   If the lower left is $(-\infty, -\infty)$ then this means

$$\lim_{(x,y)\to(-\infty,-\infty)} F(x, y) = 0.$$

3. $F(x, y) = 1$ at the upper right of its range.

# Marginal pmf

Roll two dice: $X = \#$ on first die, $T =$ total on both dice.

The marginal pmf of $X$ is found by summing the rows. The marginal pmf of $T$ is found by summing the columns

| $X \backslash T$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | $p(x_i)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 0 | 0 | 0 | 0 | 0 | 1/6 |
| 2 | 0 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 0 | 0 | 0 | 0 | 1/6 |
| 3 | 0 | 0 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 0 | 0 | 0 | 1/6 |
| 4 | 0 | 0 | 0 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 0 | 0 | 1/6 |
| 5 | 0 | 0 | 0 | 0 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 0 | 1/6 |
| 6 | 0 | 0 | 0 | 0 | 0 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/6 |
| $p(t_j)$ | 1/36 | 2/36 | 3/36 | 4/36 | 5/36 | 6/36 | 5/36 | 4/36 | 3/36 | 2/36 | 1/36 | 1 |

# Marginal pdf

**Example.** Suppose $X$ and $Y$ take values on the square $[0,1] \times [1,2]$ with joint pdf $f(x,y) = \frac{8}{3}x^3y$.

The marginal pdf $f_X(x)$ is found by *integrating out* the $y$. Likewise for $f_Y(y)$.

<u>**answer:**</u>

$$f_X(x) = \int_1^2 \frac{8}{3}x^3y\,dy = \left[\frac{4}{3}x^3y^2\right]_1^2 = \boxed{4x^3}$$

$$f_Y(y) = \int_0^1 \frac{8}{3}x^3y\,dx = \left[\frac{2}{3}x^4y^1\right]_0^1 = \boxed{\frac{2}{3}y}.$$

# Board question

Suppose $X$ and $Y$ are random variables and

- $(X, Y)$ takes values in $[0, 1] \times [0, 1]$.
- the pdf is $\frac{3}{2}(x^2 + y^2)$.

1. Show $f(x, y)$ is a valid pdf.
2. Visualize the event $A =$ '$X > .3$ and $Y > .5$'. Find its probability.
3. Find the cdf $F(x, y)$.
4. Find the marginal pdf $f_X(x)$. Use this to find $P(X < .5)$.
5. Use the cdf $F(x, y)$ to find the marginal cdf $F_X(x)$ and $P(X < .5)$.
6. See next slide

# Board question continued

6. (New scenario) From the following table compute $F(3.5, 4)$.

| $X\backslash Y$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 |
| 2 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 |
| 3 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 |
| 4 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 |
| 5 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 |
| 6 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 |

# Independence

Events $A$ and $B$ are independent if

$$P(A \cap B) = P(A)P(B).$$

Random variables $X$ and $Y$ are independent if

$$F(x, y) = F_X(x)F_Y(y).$$

Discrete random variables $X$ and $Y$ are independent if

$$p(x_i, y_j) = p_X(x_i)p_Y(y_j).$$

Continuous random variables $X$ and $Y$ are independent if

$$f(x, y) = f_X(x)f_Y(y).$$

# Concept question: independence I

Roll two dice: $X$ = value on first, $Y$ = value on second

| $X \backslash Y$ | 1 | 2 | 3 | 4 | 5 | 6 | $p(x_i)$ |
|---|---|---|---|---|---|---|---|
| 1 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/6 |
| 2 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/6 |
| 3 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/6 |
| 4 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/6 |
| 5 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/6 |
| 6 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/6 |
| $p(y_j)$ | 1/6 | 1/6 | 1/6 | 1/6 | 1/6 | 1/6 | 1 |

Are $X$ and $Y$ independent? 1. Yes 2. No

# Concept question: independence II

Roll two dice: $X$ = value on first, $T$ = sum

| $X\backslash T$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | $p(x_i)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 0 | 0 | 0 | 0 | 0 | 1/6 |
| 2 | 0 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 0 | 0 | 0 | 0 | 1/6 |
| 3 | 0 | 0 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 0 | 0 | 0 | 1/6 |
| 4 | 0 | 0 | 0 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 0 | 0 | 1/6 |
| 5 | 0 | 0 | 0 | 0 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 0 | 1/6 |
| 6 | 0 | 0 | 0 | 0 | 0 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/6 |
| $p(y_j)$ | 1/36 | 2/36 | 3/36 | 4/36 | 5/36 | 6/36 | 5/36 | 4/36 | 3/36 | 2/36 | 1/36 | 1 |

Are $X$ and $Y$ independent? 1. Yes 2. No

# Concept Question

Among the following pdf's which are independent? (Each of the ranges is a rectangle chosen so that $\int\int f(x,y)\,dx\,dy = 1$.)

i) $f(x,y) = 4x^2y^3$.

ii) $f(x,y) = \frac{1}{2}(x^3y + xy^3)$.

iii) $f(x,y) = 6e^{-3x-2y}$

Put a 1 for independent and a 0 for not-independent.

(a) 111 (b) 110 (c) 101 (d) 100

(e) 011 (f) 010 (g) 001 (h) 000

# Covariance

Measures the degree to which two random variables vary together, e.g. height and weight of people.

$X$, $Y$ random variables with means $\mu_X$ and $\mu_Y$

$$\text{Cov}(X, Y) = E((X - \mu_X)(Y - \mu_Y)).$$

# Properties of covariance

Properties

1. $\text{Cov}(aX + b, cY + d) = ac\text{Cov}(X, Y)$ for constants $a, b, c, d$.
2. $\text{Cov}(X_1 + X_2, Y) = \text{Cov}(X_1, Y) + \text{Cov}(X_2, Y)$.
3. $\text{Cov}(X, X) = \text{Var}(X)$
4. $\text{Cov}(X, Y) = E(XY) - \mu_X \mu_Y$.
5. If $X$ and $Y$ are independent then $\text{Cov}(X, Y) = 0$.
6. **Warning:** The converse is not true, if covariance is 0 the variables might not be independent.

# Concept question

Suppose we have the following joint probability table.

| $Y \backslash X$ | -1 | 0 | 1 | $p(y_j)$ |
|---|---|---|---|---|
| 0 | 0 | 1/2 | 0 | 1/2 |
| 1 | 1/4 | 0 | 1/4 | 1/2 |
| $p(x_i)$ | 1/4 | 1/2 | 1/4 | 1 |

We compute the covariance using the following steps.

1. $E(X) = 0 \quad E(Y) = 1/2$
2. $E(XY) = -1/4 + 1/4 = 0$
3. Therefore $\text{Cov}(X, Y) = E(XY) - E(X)E(Y) = 0$.

Because the covariance is 0 we know that $X$ and $Y$ are independent

1. True   2. False

# Concept question

Suppose we have the following joint probability table.

| $Y \backslash X$ | -1 | 0 | 1 | $p(y_j)$ |
|---|---|---|---|---|
| 0 | 0 | 1/2 | 0 | 1/2 |
| 1 | 1/4 | 0 | 1/4 | 1/2 |
| $p(x_i)$ | 1/4 | 1/2 | 1/4 | 1 |

We compute the covariance using the following steps.

1. $E(X) = 0 \quad E(Y) = 1/2$
2. $E(XY) = -1/4 + 1/4 = 0$
3. Therefore $\text{Cov}(X, Y) = E(XY) - E(X)E(Y) = 0$.

Because the covariance is 0 we know that $X$ and $Y$ are independent

1. True  2. False

Key point: covariance measures the linear relationship between $X$ and $Y$. It can completely miss a quadratic or higher order relationship.

# Board question: computing covariance

Flip a fair coin 3 times.

Let $X =$ number of heads in the first 2 flips

Let $Y =$ number of heads on the last 2 flips.

Compute $\text{Cov}(X, Y)$,

# Stop and rest

We'll stop here and finish the remainder of these slides in the next class.

# Correlation

Like covariance, but removes scale.
The *correlation coefficient* between $X$ and $Y$ is defined by

$$\text{Cor}(X, Y) = \rho = \frac{\text{Cov}(X, Y)}{\sigma_X \, \sigma_Y}.$$

Properties:
1. $\rho$ is the covariance of the standardized versions of $X$ and $Y$.
2. $\rho$ is dimensionless (it's a ratio).
3. $-1 \leq \rho \leq 1$. $\rho = 1$ if and only if $Y = aX + b$ with $a > 0$ and $\rho = -1$ if and only if $Y = aX + b$ with $a < 0$.

# Real-life correlations

- Over time, amount of Ice cream consumption is correlated with number of pool drownings.
- In 1685 (and today) being a student is the most dangerous profession.
- In 90% of bar fights ending in a death the person who started the fight died.
- Hormone replacement therapy (HRT) is correlated with a lower rate of coronary heart disease (CHD).

# Correlation is not causation

Edward Tufte: "Empirically observed covariation is a necessary but not sufficient condition for causality."

# Overlapping sums of uniform random variables

We made two random variables $X$ and $Y$ from overlapping sums of uniform random variables

For example:

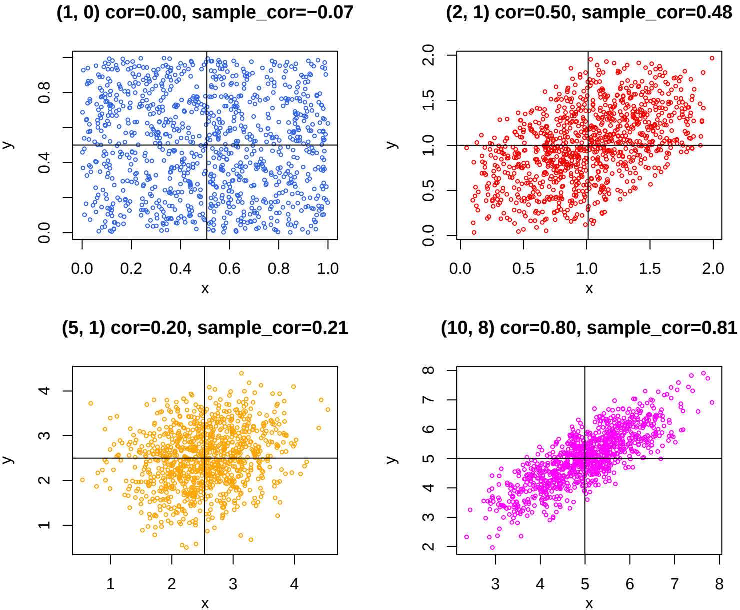
$$X = X_1 + X_2 + X_3 + X_4 + X_5$$
$$Y = X_3 + X_4 + X_5 + X_6 + X_7$$

These are sums of 5 of the $X_i$ with 3 in common.

If we sum $r$ of the $X_i$ with $s$ in common we name it $(r, s)$.

Below are a series of scatterplots produced using R.

# Scatter plots



# Concept question

Toss a fair coin $2n + 1$ times. Let $X$ be the number of heads on the first $n + 1$ tosses and $Y$ the number on the last $n + 1$ tosses.

If $n = 1000$ then $\text{Cov}(X, Y)$ is:

(a) 0 (b) $1/4$ (c) $1/2$ (d) 1

(e) More than 1 (f) tiny but not 0

# Board question

Toss a fair coin $2n+1$ times. Let $X$ be the number of heads on the first $n+1$ tosses and $Y$ the number on the last $n+1$ tosses.

Compute $\text{Cov}(X, Y)$ and $\text{Cor}(X, Y)$.