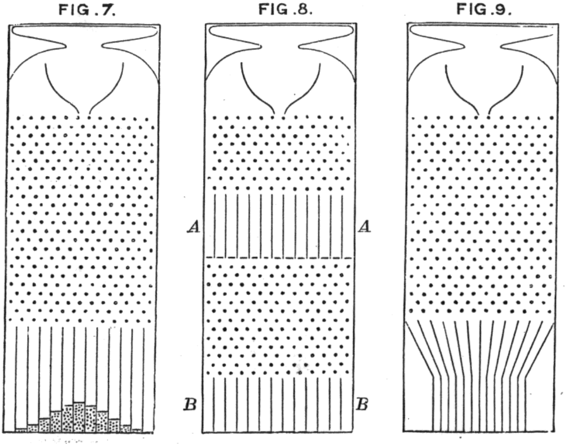
Discrete Random Variables; Expectation

18.05 Spring 2014

Jeremy Orloff and Jonathan Bloom



http://www.mathsisfun.com/data/quincunx.html

http://www.youtube.com/watch?v=9xUBhhM4vbM

Board Question: Evil Squirrels

One million squirrels, of which 100 are pure evil!

Events: E = evil, G = good, A = alarm sounds

Accuracy: $P(A|E) = .99$, $P(A|G) = .01$.

a) A squirrel sets off the alarm, what is the probability that it is evil?

b) Is the system of practical use?

answer: a) Let E be the event that a squirrel is evil. Let A be the event that the alarm goes off. By Bayes Theorem, we have:

$$\begin{aligned} P(E\,|\,A) &= \frac{P(A\,|\,E)P(E)}{P(A\,|\,E)P(E) + P(A\,|\,E^c)P(E^c)} \\ &= \frac{.99\frac{100}{1000000}}{.99\frac{100}{1000000} + .01\frac{999900}{1000000}} \\ &\approx .01. \end{aligned}$$

b) No. The alarm would be more trouble than its worth, since for every true positive there are about 99 false positives.

Big point

	Evil	Nice	
Alarm	99	9999	10098
No alarm	1	989901	989902
	100	999900	1000000

Summary:

$$\text{Probability a random test is correct} = \frac{99+989901}{1000000} = .99$$

$$\text{Probability a positive test is correct} = \frac{99}{10098} \approx .01$$

These probabilities are not the same!

Table Question: Dice Game

1. The Randomizer holds the 6-sided die in one fist and the 8-sided die in the other.
2. The Roller selects one of the Randomizer's fists and covertly takes the die.
3. The Roller rolls the die in secret and reports the result to the table.

Given the reported number, what is the probability that the 6-sided die was chosen?

Reading Review

Random variable X assigns a number to each outcome:

$$X : \Omega \to \mathbf{R}$$

"$X = a$" denotes the event $\{\omega \,|\, X(\omega) = a\}$.

Probability mass function (pmf) of X is given by

$$p(a) = P(X = a).$$

Cumulative distribution function (cdf) of X is given by

$$F(a) = P(X \leq a).$$

Concept Question: cdf and pmf

X a random variable.

values of X:	1	3	5	7
cdf $F(a)$:	.5	.75	.9	1

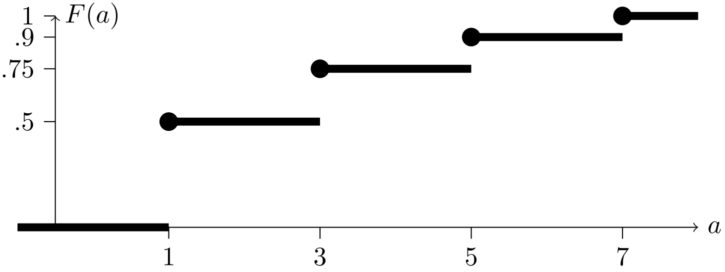
1. What is $P(X \leq 3)$?

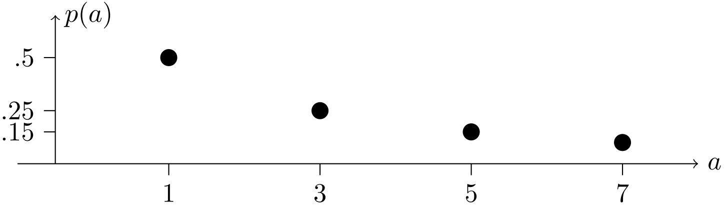
a) .15 b) .25 c) .5 d) .75

2. What is $P(X = 3)$

a) .15 b) .25 c) .5 d) .75

CDF and PMF





Deluge of discrete distributions

Bernoulli(p) = 1 (success) with probability p,
0 (failure) with probability $1 - p$.

In more neutral language:
Bernoulli(p) = 1 (heads) with probability p,
0 (tails) with probability $1 - p$.

Binomial(n,p) = # of successes in n independent Bernoulli(p) trials.

Geometric(p) = # of tails before first heads in a sequence of indep. Bernoulli(p) trials.
(Neutral language avoids confusing whether we want the number of successes before the first failure or vice versa.)

Concept Question

1\. Let $X \sim \text{binom}(n, p)$ and $Y \sim \text{binom}(m, p)$ be independent. Then $X + Y$ follows:

a) $\text{binom}(n + m, p)$ b) $\text{binom}(nm, p)$

c) $\text{binom}(n + m, 2p)$ d) other

2\. Let $X \sim \text{binom}(n, p)$ and $Z \sim \text{binom}(n, q)$ be independent. Then $X + Z$ follows:

a) $\text{binom}(n, p + q)$ b) $\text{binom}(n, pq)$

c) $\text{binom}(2n, p + q)$ d) other

Board Question: Find the pmf

$X = \#$ of successes before the *second* failure of a sequence of independent Bernoulli(p) trials.

Describe the pmf of X.

Dice simulation: geometric(1/4)

Roll the 4-sided die repeatedly until you roll a 1.
Click in $X = \#$ of rolls *BEFORE* the 1.
(If X is 9 or more click 9.)

Example: If you roll (3, 4, 2, 3, 1) then click in 4.
Example: If you roll (1) then click 0.

Fiction

Gambler's fallacy: [roulette] if black comes up several times in a row then the next spin is more likely to be red.

Hot hand: NBA players get 'hot'.

Fact

P(red) remains the same.

The roulette wheel has no memory. (Monte Carlo, 1913).

The data show that player who has made 5 shots in a row is no more likely than usual to make the next shot.

Memory

Show that *Geometric*(p) is memoryless, i.e.

$$P(X = n + k \mid X \geq n) = P(X = k)$$

Explain why we call this memoryless.

Computing expected value

Definition: $$E(X) = \sum_i x_i\, p(x_i)$$

1. $E(aX + b) = aE(X) + b$
2. $E(X + Y) = E(X) + E(Y)$
3. $E(h(X)) = \sum_i h(x_i)\, p(x_i)$

Board Question: Interpreting Expectation

a) Would you accept a gamble that offers a 10% chance to win \$95 and a 90% chance of losing \$5?

b) Would you pay \$5 to participate in a lottery that offers a 10% percent chance to win \$100 and a 90% chance to win nothing?

- Find the expected value of your change in assets in each case?

Board Question

Suppose (hypothetically!) that everyone at your table got up, ran around the room, and sat back down randomly (i.e., all seating arrangements are equally likely).

What is the expected value of the number of people sitting in their original seat?

(We will explore this with simulations in Friday Studio.)