Probability Review for Final Exam

18.05 Spring 2014

Jeremy Orloff and Jonathan Bloom

Problem 1. We compute

$$E[X] = -2 \cdot \frac{1}{15} + -1 \cdot \frac{2}{15} + 0 \cdot \frac{3]15 + 1 \cdot \frac{[}{4}}{15} + 2 \cdot \frac{5}{15} = \frac{2}{3}.$$

Thus

$$\text{Var}(X) = E((X - \frac{2}{3})^2) = \frac{14}{9}.$$

Problem 2. We first compute

$$E[X] = \int_0^1 x \cdot 2x dx = \frac{2}{3}$$

$$E[X^2] = \int_0^1 x^2 \cdot 2x dx = \frac{1}{2}$$

$$E[X^4] = \int_0^1 x^4 \cdot 2x dx = \frac{1}{3}.$$

Thus,

$$\text{Var}(X) = E[X^2] - (E[X])^2 = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}$$

and

$$\text{Var}(X^2) = E[X^4] = \left(E[X^2]\right)^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.$$

Problem 3. Use $\text{Var}(X) = E(X^2) - E(X)^2 \;\Rightarrow\; 3 = E(X^2) - 4 \;\Rightarrow\; \boxed{E(X^2) = 7.}$

Problem 4. **answer:**

Make a table

X:	0	1
prob:	(1-p)	p
X^2	0	1.

From the table, $\boxed{E(X) = 0 \cdot (1-p) + 1 \cdot p = p.}$

Since X and X^2 have the same table $E(X^2) = E(X) = p$.

Therefore, $\boxed{\text{Var}(X) = p - p^2 = p(1-p).}$

Problem 5. Let X be the number of people who get their own hat.

Following the hint: let X_j represent whether person j gets their own hat. That is,

$X_j = 1$ if person j gets their hat and 0 if not.

We have, $X = \sum_{j=1}^{100} X_j$, so $E(X) = \sum_{j=1}^{100} E(X_j)$.

Since person j is equally likely to get any hat, we have $P(X_j = 1) = 1/100$. Thus, $X_j \sim \text{Bernoulli}(1/100) \;\Rightarrow\; E(X_j) = 1/100 \;\Rightarrow\; \boxed{E(X) = 1.}$

Problem 6. **(a)** There are a number of ways to present this.

$X \sim 3\,\text{binomial}(25, 1/6)$, so

$$P(X = 3k) = \binom{25}{k}\left(\frac{1}{6}\right)^k\left(\frac{5}{6}\right)^{25-k}, \quad \text{for } k = 0,\, 1,\, 2,\, \ldots,\, 25.$$

(b) $X \sim 3\,\text{binomial}(25, 1/6)$.

Recall that the mean and variance of binomial(n, p) are np and $np(1-p)$. So,

$$E(X) = 3\,E(textbinomial(25, 1/6)) = 3{\cdot}25/6 = 75/6, \text{ and } \text{Var}(X) = 9\,\text{Var}(textbinomial(25, 1/6)) = 9{\cdot}25($$

(c) $E(X+Y) = E(X) + E(Y) = 150/6 = 25.$, $E(2X) = 2E(X) = 150/6 = 25.$

$\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y) = 250/4$. $\text{Var}(2X) = 4\text{Var}(X) = 500/4$.

The means of $X+Y$ and $2X$ are the same, but $\text{Var}(2X) > \text{Var}(X+Y)$.

This makes sense because in $X+Y$ sometimes X and Y will be on opposite sides from the mean so distances to the mean will tend to cancel, However in $2X$ the distance to the mean is always doubled.

Problem 7. First we find the value of a:

$$\int_0^1 f(x)\,dx = 1 = \int_0^1 x + ax^2\,dx = \frac{1}{2} + \frac{a}{3} \;\Rightarrow\; a = 3/2.$$

The CDF is $F_X(x) = P(X \le x)$. We break this into cases:

(i) $b < 0 \;\Rightarrow\; F_X(b) = 0$.

(ii) $0 \le b \le 1 \;\Rightarrow\; F_X(b) = \int_0^b x + \frac{3}{2}x^2\,dx = \frac{b^2}{2} + \frac{b^3}{2}$.

(iii) $1 < x \;\Rightarrow\; F_X(b) = 1$.

Using F_X we get

$$P(.5 < X < 1) = F_X(1) - F_X(.5) = 1 - \left(\frac{.5^2 + .5^3}{2}\right) = \frac{13}{16}.$$

Problem 8. (i) yes, discrete, (ii) no, (iii) no, (iv) no, (v) yes, continuous

(vi) no (vii) yes, continuous, (viii) yes, continuous.

Problem 9. **(a)** We compute

$$P(X \geq 5) = 1 - P(X < 5) = 1 - \int_0^5 \lambda e^{-\lambda x} dx = 1 - (1 - e^{-5\lambda}) = e^{-5\lambda}.$$

(b) We want $P(X \geq 15 | X \geq 10)$. First observe that $P(X \geq 15, X \geq 10) = P(X \geq 15)$. From similar computations in (a), we know

$$P(X \geq 15) = e^{-15\lambda} \qquad\qquad P(X \geq 10) = e^{-10\lambda}.$$

From the definition of conditional probability,

$$P(X \geq 15 | X \geq 10) = \frac{P(X \geq 15, X \geq 10)}{P(X \geq 10)} = \frac{P(X \geq 15)}{P(X \geq 10)} = e^{-5\lambda}$$

Note: This is an illustration of the memorylessness property of the exponential distribution.

Problem 10.

(a) We did this in class. Let $\phi(z)$ and $\Phi(z)$ be the PDF and CDF of Z.
$F_Y(y) = P(Y \leq y) = P(aZ + b \leq y) = P(Z \leq (y-b)/a) = \Phi((y-b)/a)$.
Differentiating:

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}\Phi((y-b)/a) = \frac{1}{a}\phi((y-b)/a) = \frac{1}{\sqrt{2\pi}\, a} e^{-(y-b)^2/2a^2}.$$

Since this is the density for $\mathrm{N}(b, a^2)$ we have shown $Y \sim \mathrm{N}(b, a^2)$.

(b) By part (a), $Y \sim \mathrm{N}(\mu, \sigma^2) \Rightarrow Y = \sigma Z + \mu$.
But, this implies $(Y - \mu)/\sigma = Z \sim \mathrm{N}(0, 1)$. QED

Problem 11. **(a)** $E(W) = 3E(X) - 2E(Y) + 1 = 6 - 10 + 1 = \boxed{-3}$
$\mathrm{Var}(W) = 9\mathrm{Var}(X) + 4\mathrm{Var}(Y) = 45 + 36 = \boxed{81}$

(b) Since the sum of independent normal is normal part (a) shows: $W \sim N(-3, 81)$.
Let $Z \sim N(0, 1)$. We *standardize* W: $P(W \leq 6) = P\left(\frac{W+3}{9} \leq \frac{9}{9}\right) = P(Z \leq 1) \approx \boxed{.84.}$

Problem 12.

Method 1

$U(a, b)$ has density $f(x) = \dfrac{1}{b-a}$ on $[a, b]$. So,

$$E(X) = \int_a^b x f(x)\,dx = \frac{1}{b-a}\int_a^b x\,dx = \left.\frac{x^2}{2(b-a)}\right|_a^b = \frac{b^2-a^2}{2(b-a)} = \boxed{\frac{a+b}{2}}.$$

$$E(X^2) = \int_a^b x^2 f(x)\,dx = \frac{1}{b-a}\int_a^b x^2\,dx = \left.\frac{x^3}{3(b-a)}\right|_a^b = \frac{b^3-a^3}{3(b-a)}.$$

Finding $\text{Var}(X)$ now requires a little algebra,

$$\text{Var}(X) = E(X^2) - E(X)^2 = \frac{b^3-a^3}{3(b-a)} - \frac{(b+a)^2}{4}$$

$$= \frac{4(b^3-a^3) - 3(b-a)(b+a)^2}{12(b-a)} = \frac{b^3 - 3ab^2 + 3a^2b - a^3}{12(b-a)} = \frac{(b-a)^3}{12(b-a)} = \boxed{\frac{(b-a)^2}{12}}.$$

Method 2

There is an easier way to find $E(X)$ and $\text{Var}(X)$.

Let $U \sim \text{U}(a, b)$. Then the calculations above show $E(U) = 1/2$ and $(E(U^2) = 1/3$ $\Rightarrow$ $\text{Var}(U) = 1/3 - 1/4 = 1/12$.

Now, we know $X = (b-a)U + a$, so $E(X) = (b-a)E(U) + a = (b-a)/2 + a = (b+a)/2$ and $\text{Var}(X) = (b-a)^2\text{Var}(U) = (b-a)^2/12$.

Problem 13.

(a) $S_n \sim \text{Binomial}(n,\ p)$, since it is the number of successes in n independent Bernoulli trials.

(b) $T_m \sim \text{Binomial}(m,\ p)$, since it is the number of successes in m independent Bernoulli trials.

(c) $S_n + T_m \sim \text{Binomial}(n+m,\ p)$, since it is the number of successes in $n+m$ independent Bernoulli trials.

(d) Yes, S_n and T_m are independent. We haven't given a formal definition of independent random variables yet. But, we know it means that knowing S_n gives no information about T_m. This is clear since the first n trials are independent of the last m.

Problem 14. Compute the median for the exponential distribution with parameter λ. The density for this distribution is $f(x) = \lambda\,\mathrm{e}^{-\lambda x}$. We know (or can compute) that the distribution function is $F(a) = 1 - \mathrm{e}^{-\lambda a}$. The median is the value of a such that $F(a) = .5$. Thus, $1 - \mathrm{e}^{-\lambda a} = 0.5 \Rightarrow 0.5 = \mathrm{e}^{-\lambda a} \Rightarrow \log(0.5) = -\lambda a \Rightarrow$ $\boxed{a = \log(2)/\lambda.}$

Problem 15. **(a)** The joint distribution is given by

$Y \setminus X$	1	2	3	
1	$\frac{1168}{5383}$	$\frac{825}{5383}$	$\frac{305}{5383}$	$\frac{2298}{5383}$
2	$\frac{573}{5383}$	$\frac{1312}{5383}$	$\frac{1200}{5383}$	$\frac{3085}{5383}$
	$\frac{1741}{5383}$	$\frac{2137}{5383}$	$\frac{1505}{5383}$	1

with the marginal distribution of X at right and of Y at bottom.

(b) X and Y are dependent because, for example,

$$P(X = 1 \text{ and } Y = 1) = \frac{1168}{5383}$$

is not equal to

$$P(X = 1)P(Y = 1) = \frac{1741}{5383} \cdot \frac{2298}{5383}.$$

Problem 16. **(a)** Here we have two continuous random variables X and Y with going potability density function

$$f(x, y) = \frac{12}{5}xy(1 + y) \quad \text{for } 0 \le x \le 1 \text{ and } 0 \le y \le 1,$$

and $f(x, y) = 0$ otherwise. So

$$P(\frac{1}{4} \le X \le \frac{1}{2}, \frac{1}{3} \le Y \le \frac{2}{3}) = \int_{\frac{1}{4}}^{\frac{1}{2}} \int_{\frac{1}{3}}^{\frac{2}{3}} f(x, y)dy\, dx = \frac{41}{720}.$$

(b) $F(a, b) = \int_0^a \int_0^b f(x, y)dy\, dx = \frac{3}{5}a^2b^2 + \frac{2}{5}a^2b^3$ for $0 \le a \le 1$ and $0 \le b \le 1$.

(c) Since $f(x, y) = 0$ for $y > 1$, we have

$$F_X(a) = \lim_{b\to\infty} F(a, b) = F(a, 1) = a^2.$$

(d) For $0 \le x \le 1$, we have

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)dy = \int_0^1 f(x, y)dy = 2x.$$

This is consistent with (c) because $\frac{d}{dx}(x^2) = 2x$.

(e) We first compute $f_Y(y)$ for $0 \le y \le 1$ as

$$f_Y(y) = \int_0^1 f(x, y)dx = \frac{6}{5}y(y + 1).$$

Since $f(x,y) = f_X(x)f_Y(y)$, we conclude that X and Y are independent.

Problem 17. **(a)** The marginal probability $P_Y(1) = 1/2$
$\Rightarrow P(X = 0, Y = 1) = P(X = 2, Y = 1) = 0$.
Now each column has one empty entry. This can be computed by making the column add up to the given marginal probability.

$Y\backslash X$	0	1	2	P_Y
-1	1/6	1/6	1/6	1/2
1	0	1/2	0	1/2
P_X	1/6	2/3	1/6	1

(b) No, X and Y are not independent.
For example, $P(X = 0, Y = 1) = 0 \neq 1/12 = P(X = 0) \cdot P(Y = 1)$.

Problem 18. For shorthand, let $P(X = a, Y = b) = p(a,b)$.
(a) $P(X = Y) = p(1,1) + p(2,2) + p(3,3) + p(4,4) = \boxed{34/136.}$
(b) $P(X + Y = 5) = p(1,4) + p(2,3) + p(3,2) + p(4,1) = \boxed{34/136.}$
(c) $P(1 < X \leq 3, 1 < Y \leq 3) =$ sum of middle 4 probabilities in table $=\boxed{34/136.}$
(d) $\{1,4\} \times \{1,4\} = \{(1,1),(1,4),(4,1),(4,4)\} \Rightarrow$ prob. $= \boxed{34/136.}$

Problem 19. **(a)** X and Y are independent, so the table is computed from the product of the known marginal probabilities. Since they are independent, $\text{Cov}(X,Y) = 0$.

$Y\backslash X$	0	1	P_Y
0	1/8	1/8	1/4
1	1/4	1/4	1/2
2	1/8	1/8	1/4
P_X	1/2	1/2	1

(b) The sample space is $\Omega = \{$HHH, HHT, HTH, HTT, THH, THT, TTH, TTT$\}$.
$P(X = 0, Z = 0) = P(\{TTH, TTT\}) = 1/4$.
$P(X = 0, Z = 1) = P(\{THH, THT\}) = 1/4$.
$P(X = 0, Z = 2) = 0$.
$P(X = 1, Z = 0) = 0$.
$P(X = 1, Z = 1) = P(\{HTH, HTT\}) = 1/4$.
$P(X = 1, Z = 2) = P(\{HHH, HHT\}) = 1/4$.

$Z\backslash X$	0	1	P_Z
0	1/4	0	1/4
1	1/4	1/4	1/2
2	0	1/4	1/4
P_X	1/2	1/2	1

$\text{Cov}(X,Z) = E(XZ) - E(X)E(Z)$.
$E(X) = 1/2,\quad E(Z) = 1,\ E(XZ) = \sum x_i y_j p(x_i, y_j) = 3/4$.
$\Rightarrow \text{Cov}(X,Z) = 3/4 - 1/2 = \boxed{1/4.}$

Problem 20. **(a)** $F(a,b) = P(X \leq a, Y \leq b) = \displaystyle\int_0^a \int_0^b (x+y)\, dy\, dx.$

Inner integral: $xy + \left.\frac{y^2}{2}\right|_0^b = xb + \frac{b^2}{2}$. Outer integral: $\left.\frac{x^2}{2}b + \frac{b^2}{2}x\right|_0^a = \frac{a^2b + ab^2}{2}$.

So $\boxed{F(x,y) = \frac{x^2y + xy^2}{2}}$ and $\boxed{F(1,1) = 1.}$

(b) $f_X(x) = \int_0^1 f(x,y)\,dy = \int_0^1 (x+y)\,dy = xy + \left.\frac{y^2}{2}\right|_0^1 = \boxed{x + \frac{1}{2}.}$

By symmetry, $\boxed{f_Y(y) = y + 1/2.}$

(c) To see if they are independent we check if the joint density is the product of the marginal densities.

$f(x,y) = x + y, \quad f_X(x) \cdot f_Y(y) = (x + 1/2)(y + 1/2).$

Since these are not equal, $\boxed{X \text{ and } Y \text{ are not independent.}}$

(d) $E(X) = \int_0^1 \int_0^1 x(x+y)\,dy\,dx = \int_0^1 \left[x^2y + x\left.\frac{y^2}{2}\right|_0^1 \right] dx = \int_0^1 x^2 + \frac{x}{2}\,dx = \boxed{\frac{7}{12}.}$

(Or, using (b), $E(X) = \int_0^1 x f_X(x)\,dx = \int_0^1 x(x + 1/2)\,dx = 7/12$.)

By symmetry $\boxed{E(Y) = 7/12.}$

$E(X^2 + Y^2) = \int_0^1 \int_0^1 (x^2 + y^2)(x+y)\,dy\,dx = \boxed{\frac{5}{6}.}$

$E(XY) = \int_0^1 \int_0^1 xy(x+y)\,dy\,dx = \boxed{\frac{1}{3}.}$

$\text{Cov}(X,Y) = E(XY) - E(X)E(Y) = \frac{1}{3} - \frac{49}{144} = \boxed{-\frac{1}{144}.}$

Problem 21.
Standardize:

$$\begin{aligned} P\left(\sum_i X_i < 30\right) &= P\left(\frac{\frac{1}{n}\sum X_i - \mu}{\sigma/\sqrt{n}} < \frac{30/n - \mu}{\sigma/\sqrt{n}}\right) \\ &\approx P\left(Z < \frac{30/100 - 1/5}{1/30}\right) \quad \text{(by the central limit theorem)} \\ &= P(Z < 3) \\ &= 1 - .0013 = .9987 \ \text{(from the table)} \end{aligned}$$

Problem 22. Let $\overline{X} = \frac{X_1 + \ldots + X_{144}}{144} \Rightarrow E(\overline{X}) = 2$, and $\sigma_{\overline{X}} = 2/12 = 1/6$.
($\sqrt{n} = 12$)

A chain of algebra gives $P(X_1 + \ldots + X_{144}) + P\left(\overline{X} > \frac{264}{144}\right) = P\left(\overline{X} > 1.8333\right)$.

Standardization gives $P(\overline{X} > 1.8333) = P\left(\frac{\overline{X}-2}{1/6} > \frac{1.8333-2}{1/6}\right) = P\left(\frac{\overline{X}-2}{1/6} > -1.0\right)$

Now, the Central limit theorem says $P\left(\frac{\overline{X}-2}{1/6} > -1.0\right) \approx P(Z > -1) = \boxed{.84}$

Problem 23. Let X_j be the IQ of a randomly selected person. We are given $E(X_j) = 100$ and $\sigma_{X_j} = 15$.

Let $\overline{X}$ be the average of the IQ's of 100 randomly selected people. We have $(\overline{X}) = 100$ and $\sigma_{\overline{X}} = 15/\sqrt{100} = 1.5$.

The problem asks for $P(\overline{X} > 115)$. Standardizing we get $P(\overline{X} > 115) \approx P(Z > 10)$. $\boxed{\text{This is effectively 0.}}$

Problem 24. Data mean and variance $\bar{x} = 65$, $s^2 = 35.778$. The number of degrees of freedom is 9. We look up $t_{9,.025} = 2.262$ in the t-table The 95% confidence interval is

$$\left[\bar{x} - \frac{t_{9,.025}s}{\sqrt{n}}, \bar{x} + \frac{t_{9,.025}s}{\sqrt{n}}\right] = \left[65 - 2.262\sqrt{3.5778}, 65 + 2.262\sqrt{3.5778}\right] = [60.721, 69.279]$$

Problem 25. Suppose we have taken data $x_1, \ldots, x_n$ with mean $\bar{x}$. Remember in these probabilities μ is a given (fixed) hypothesis.

$$P(|\bar{x}-\mu| \le .5 \mid \mu) = .95 \Leftrightarrow P\left(\frac{|\bar{x}-\mu|}{\sigma/\sqrt{n}} < \frac{.5}{\sigma/\sqrt{n}} \mid \mu\right) = .95 \Leftrightarrow P\left(|Z| < \frac{.5\sqrt{n}}{5}\right) = .95.$$

Using the table, we have precisely that $\frac{.5\sqrt{n}}{5} = 1.96$. So, $n = (19.6)^2 = \boxed{384.}$.

If we use our rule of thumb that the .95 interval is 2σ we have $\sqrt{n}/10 = 2 \Rightarrow n = 400$.

Problem 26. The rule-of-thumb is that a 95% confidence interval is $\bar{x} \pm 1/\sqrt{n}$. To be within 1% we need

$$\frac{1}{\sqrt{n}} = .01 \Rightarrow n = 10000.$$

Using $z_{.025} = 1.96$ instead the 95% confidence interval is

$$\bar{x} \pm \frac{z_{.025}}{2\sqrt{n}}.$$

To be within 1% we need

$$\frac{z_{.025}}{2\sqrt{n}} = .01 \Rightarrow n = 9604.$$

Note, we are using the standard Bernoulli approximation $\sigma \le 1/2$.

Problem 27. The 90% confidence interval is

$$\overline{x} \pm \frac{z_{.05}}{2\sqrt{n}} = \overline{x} \pm \frac{1.64}{40}$$

We want $\overline{x} - \frac{1.64}{40} > .5$, that is $\overline{x} > .541$.

So $\frac{\text{number preferring A}}{400} > .541$. So,

$$\text{number preferring A} > 216.4$$

Problem 28. A 95% confidence means about 5% = 1/20 will be wrong. You'd expect about 2 to be wrong.

With a probability $p = .05$ of being wrong, the number wrong follows a Binomial$(40, p)$ distribution. This has expected value 2, and standard deviation $\sqrt{40(.05)(.95)} = 1.38$. 10 wrong is (10-2)/1.38 = 5.8 standard deviations from the mean. This would be surprising.

Problem 29. We have $n = 27$ and $s^2 = 5.86$. If we fix a hypothesis for σ^2 we know

$$\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}$$

We used R to find the critical values. (Or use the χ^2 table.)

```
c025 = qchisq(.975,26) = 41.923
c975 = qchisq(.025,26) = 13.844
```

The 95% confidence interval for σ^2 is

$$\left[\frac{(n-1)\cdot s^2}{c_{.025}}, \frac{(n-1)\cdot s^2}{c_{.975}}\right] = \left[\frac{26\cdot 5.86}{41.923}, \frac{26\cdot 5.86}{13.844}\right] = [3.6343,\ 11.0056]$$

We can take square roots to find the 95% confidence interval for σ

$$[1.9064,\ 3.3175]$$

Problem 30. **(a)** The model is $y_i = a + bx_i + \varepsilon_i$, where ε_i is random error. We assume the errors are independent with mean 0 and the same variance for each i (homoscedastic).

The total error squared is

$$E^2 = \sum (y_i - a - bx_i)^2 = (1-a-b)^2 + (1-a-2b)^2 + (3-a-3b)^2$$

The least squares fit is given by the values of a and b which minimize E^2. We solve for them by setting the partial derivatives of E^2 with respect to a and b to 0. In R we found that $a = 1.0$, $b = 0.5$

(b) This is similar to part (a). The model is

$$y_i = ax_{i,1} + bx_{i,2} + c + \varepsilon_i$$

where the errors ε_i are independent with mean 0 and the same variance for each i (homoscedastic).

The total error squared is

$$E^2 = \sum(y_i - ax_{i,1} - bx_{i,2} - c)^2 = (3 - a - 2b - c)^2 + (5 - 2a - 3b - c)^2 + (1 - 3a - c)^2$$

The least squares fit is given by the values of a, b and c which minimize E^2. We solve for them by setting the partial derivatives of E^2 with respect to a, b and c to 0. In R we found that $a = 0.5$, $b = 1.5$, $c = -0.5$