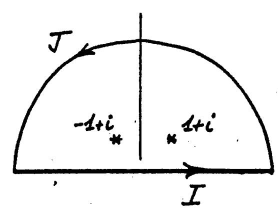
18.04 Brief Solutions to Exam #2 Fri 31 Oct 03

(1) Very briefly, the Max Modulus Principle assures us that somewhere in the vicinity of <u>any</u> "interior" location the modulus of a <u>non-constant</u> analytic function must be yet larger than "right here" ... or hence that the true maximum of $|f|$ has to lie somewhere on the boundary.

And now, since the current $|f| = 3$ everywhere on that circular boundary, it follows conversely that this same $|f| < 3$ at all points inside it. Fine and good, until one starts to reflect upon the function $g(z) = 1/f(z)$ which itself would strictly obey the MMP if $f(z) \neq 0$ at any interior location. Then likewise $|g| < \frac{1}{3}$ everywhere inside ... which leads to a big **OOPS!**

(2) $$\int_{-\infty}^{\infty} \frac{2+x^2}{4+x^4}\, dx = \boxed{\pi}$$



because our $I + J = 2\pi i \{ \underset{= -\frac{i}{4}}{\mathrm{Res}(1+i)} + \underset{=\text{ also } -\frac{i}{4}}{\mathrm{Res}(-1+i)} \}$ and plainly $J \to 0$.

(3) Much as discussed for $\int \frac{dx}{\cosh x}$ at a recent lecture, a very desirable return path for the complex integral involving $(e^z + e^{-z})^{-3}$ is at "flight-level" $y = \pi$ — meaning along $z = x + i\pi$ — since there the "East-to-West" path integral works out as a <u>simple duplicate</u> of the one we are trying to evaluate from $-\infty$ to $+\infty$ along the real axis. Thus our $2I$ here, so to speak, itself equals $2\pi i * \mathrm{Res}\{(e^z+e^{-z})^{-3};\ i\frac{\pi}{2}\}$, and our task boils down to evaluating that slightly tricky residue.

Well, since $e^z = e^{i\pi/2} e^{z - i\pi/2} = i\left[1 + (z-\frac{i\pi}{2}) + \frac{1}{2}(z-\frac{i\pi}{2})^2 + \frac{1}{6}(z-\frac{i\pi}{2})^3 + \ldots\right]$, and similarly for $e^{-z} = -i\left[1 - (\ldots) + \frac{1}{2}(\ldots)^2 - \frac{1}{6}(\ldots)^3 + \ldots\right]$, it follows that

$$e^z + e^{-z} = 2i\left[(z-\tfrac{i\pi}{2}) + \tfrac{1}{6}(z-\tfrac{i\pi}{2})^3 + \ldots\right] = 2i\,(z-\tfrac{i\pi}{2})\left[1 + \tfrac{1}{6}(z-\tfrac{i\pi}{2})^2 + \ldots\right]$$

or $$\frac{1}{(e^z+e^{-z})^3} = \frac{1}{(2i)^3}\frac{1}{(z-\frac{i\pi}{2})^3}\left[1 - \frac{3}{6}(z-\tfrac{i\pi}{2})^2 + \ldots\right].$$ Hence $\mathrm{Res} = \frac{-3/6}{(2i)^3}$,

or our $2I = 2\pi i * \mathrm{Res} = \frac{\pi}{8}$. But of course $\int_{-\infty}^{\infty} \frac{dx}{\cosh^3 x} = \underline{\underline{8}} I = \boxed{\pi/2}$.