18.04 Brief Solutions to Exam #1

Mon 06 Oct 03

① (a) Since $|\sqrt{3}-i|=2$ and $|1+i|=\sqrt{2}$, we plainly need $n=2m$ here lest the final moduli disagree. And as for the arguments, since $\arg(\sqrt{3}-i)=-\pi/6$ and $\arg(1+i)^2=\pi/2$, it seems that tripling both — to $-\pi/2$ and to $3\pi/2$ — will do fine! $\boxed{m=3,\ n=6}$

(b) Slightly rewritten, here we need $z^3=-(z+2)^3$ ... which soon argues for $\operatorname{Re} z=-1$ to remain equidistant from $z=0$ and $z=-2$. As for how "high" on that vertical locus, further sleuthing before long recommends $\boxed{z=-1 \text{ or } z=-1\pm i\sqrt{3}}$.

② (a) Well, "harmonic" of course means that $F_{xx}+F_{yy}=0$, where the subscripts abbreviate partial derivatives, whereas the real and imaginary parts of an analytic function $u+iv$ must satisfy the two CR conditions $u_x=v_y$ and $u_y=-v_x$.

Pulling these pieces together, we need to ask: $\boxed{\frac{\partial}{\partial x}F_x \overset{?}{=} \frac{\partial}{\partial y}(-F_y)}$ $\Downarrow$ $F_{xx}=-F_{yy}$ yes!

and also $\boxed{\frac{\partial}{\partial y}F_x \overset{?}{=} -\frac{\partial}{\partial x}(-F_y)} \Rightarrow F_{xy} \overset{?}{=} F_{yx}$ also yes, assuming continuity of these derivs.

(b) Similarly, if we already know that $u_x=v_y$, $u_y=-v_x$ (as for such "harmonic conjugates"), then $\boxed{\nabla^2(uv)} \overset{?}{=} \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)uv$ $= u_{xx}v+2u_xv_x+uv_{xx}+u_{yy}v+2u_yv_y+uv_{yy}$ ... which cleans up to $v_{xy}v+2v_yv_y-uu_{xy}-v_{xy}v-2v_yv_y+uu_{xy} \overset{?}{=} \boxed{0}$. QED

③ (a) This $\tanh^{-1}z$ monstrosity could either have been rederived from

$$\tanh w \equiv \frac{e^w-e^{-w}}{e^w+e^{-w}}=z$$

onward (via $e^{2w}-1=z(e^{2w}+1)$ etc.)

or else checked by plugging back into $\boxed{\tanh\left(\frac{1}{2}\log\frac{1+z}{1-z}\right) \overset{??}{=} z}$.

And, amazingly, the latter hope is indeed fulfilled, since as we just noted

$$\tanh w \equiv \frac{e^{2w}-1}{e^{2w}+1} = \frac{\frac{1+z}{1-z}-1}{\frac{1+z}{1-z}+1}, \quad \text{when } 2w=\log\frac{1+z}{1-z}.$$

(b) Directly calculated, $\boxed{\frac{d}{dz}\tanh^{-1}z} \overset{?}{=} \frac{1}{2}\frac{d}{dz}\left(\log\frac{1+z}{1-z}\right) = \dots \overset{?}{=} \boxed{\frac{1}{1-z^2}}$

after some modest huffin' & puffin'.

AT