18.04 Final Exam from Fall 2002

Wednesday, December 18, 2002 Time: 9 am - 12 noon

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Final Examination in

18.04 Complex Variables with Applications

NOTE: Students are not allowed to use any books or notes during this examination. If brought into the room, such items must not be left nearby.

Personal calculators may be used, but not shared.

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1. Use complex arithmetic (rather than your trusty calculator) to demonstrate anew that

$$z = 2\cos 36^o = e^{i\pi/5} + e^{-i\pi/5}$$

is one solution of $z^4 - 3z^2 + 1 = 0$.

2. Where can we possibly be if

(a) $z^6 + z^4 + z^2 + 1 = 0$

or if (b) $\sin z = \cos z$?

3. Somehow or other, show again that

$$\frac{d}{dz}\tan^{-1} z = \frac{1}{1+z^2} ,$$

even for complex z's .

4. Evaluate $\oint_{|z|=1} z^5 e^{2/z}\, dz$ once around that unit circle.

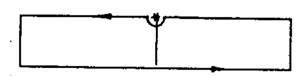
5. Let C be the ellipse $x^2/4 + y^2/9 = 1$ traversed once in the counterclockwise direction, and define

$$G(z) = \oint_C \frac{\zeta^2 - \zeta + 2}{\zeta - z}\, d\zeta \qquad \text{for any } z \text{ inside } C .$$

Then find $G(1)$, $G'(i)$ and $G''(-i)$.

6. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{x}{\sinh x}\, dx$,

HINT:



where $\sinh x = (e^x - e^{-x})/2$ as usual.

7. (a) Which region of the complex z-plane gets mapped by

$$w(z) = \frac{z-1}{z+1}$$

into the interior of the circle $|w| = 1$, and why?

(b) Use the above answer as a clue to find a related Möbius (or bilinear) transformation $W(z)$ that carries the top half of the z-plane into a unit circle centered instead at $W = 1 + i$.

8. Find that Fourier series of period 2π which represents the function

$$f(\theta;a) = \frac{1}{1 + a\cos\theta}$$

for any real a , subject only to the requirement that $|a| < 1$.

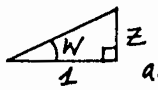
HINT: Consider terms like $p^{|k|} e^{ik\theta}$ in a complex FS.

18.04 Brief Solns to Final Exam Wed 18 Dec 02

(1) After squaring, etc: $z^4-3z^2+1 = \boxed{e^{i4\pi/5}+e^{i2\pi/5}+1+e^{-i2\pi/5}+e^{-i4\pi/5}=0}$

(2) (a) z^2 = either i or -1 or $-i$. Hence $\boxed{z = e^{\pm i\pi/4} \text{ or } e^{\pm i\pi/2} \text{ or } e^{\pm 3i\pi/2}}$

(b) Need $e^{iz}-e^{-iz} = i(e^{iz}+e^{-iz})$ or $e^{2iz}=i \to$ only $\boxed{z=\frac{\pi}{4}\pm N\pi}$ all real

(3) Well, if $w=\tan^{-1}z$, then $z=\tan w$. Hence $\frac{d}{dz}z = 1 = \frac{1}{\cos^2 w}\frac{dw}{dz}$,
or $\frac{dw}{dz}=\cos^2 w \dots$ 

$\dots\dots = 1/(1+z^2)$ QED.
as a reminder, though true even for complex w, z

(4) Since $e^{2/z}$ has Laurent series $1+\frac{2}{z}+\frac{4}{2z^2}+\frac{8}{6z^3}+\frac{16}{24z^4}+\frac{32}{120z^5}+\frac{64}{720z^6}+\dots$,
the residue at $z=0$ for $z^5e^{2/z}$ equals $64/720 \dots$ or our $\boxed{\oint \dots dz = \frac{8\pi}{45}}$

(5) Writing $f(z)=z^2-z+2$, we have from the C.I.F. (incl. derivative version) that $G(1)=2\pi i f(1) = \boxed{4\pi i}$, $G'(i)=2\pi i f'(i) = \boxed{-2\pi(2+i)}$,
and $G''(-i) = \frac{2\pi i}{2!}f''(-i) = \boxed{2\pi i}$.

(6) From solns to our Exam 3, this' $\int_{-\infty}^{\infty}\frac{x}{\sinh x}\,dx = \boxed{\frac{\pi^2}{2}}$ again.

(7) (a) Answer: right half-plane, or $\boxed{\operatorname{Re} z>0}$ ... because for all those z's — but no others — our $|\text{numerator}| = |z-1| < |\text{denominator}| = |z+1|$

(b) Clearly $p(z)=\frac{z-i}{z+i}$ will do to the upper half-plane what $w(z)$ did to the right HP. (And $q(z)=p(z)e^{i\gamma}$ will "spin" the answer plane ... but why complicate, since $\gamma=0$ will suffice.)
Still need to add $1+i$ to shift the center $\rightarrow$ $\boxed{W(z)=\frac{z-i}{z+i}e^{i\gamma}+1+i}$

(8) Taking the HINT, consider $S(\theta) = \underbrace{\dots+pe^{-i\theta}+1}_{\text{LEFT SUM}}$, $\underbrace{1+pe^{i\theta}+p^2e^{2i\theta}+p^3e^{3i\theta}+\dots}_{\text{RIGHT SUM}}$

or $S(\theta) = \dfrac{1}{1-pe^{i\theta}}+\dfrac{1}{1-pe^{-i\theta}}-1 \equiv \dfrac{1-p^2}{1+p^2-2p\cos\theta}$.

So we have a match provided $\frac{2p}{1+p^2}=a$, although then the desired FS will be $\boxed{\frac{1+p^2}{1-p^2}}$ times the sum $S(\theta)$ given above.

18.04 Final Exam from Spring 1999

Friday, May 21, 1999 Time: 9 am - 12 noon

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Final Examination in

18.04 Complex Variables with Applications

NOTE: Students are not allowed to use any books or notes during this examination. If brought into the room, such items must not be left nearby.

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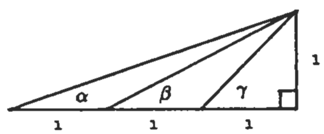
1. Use complex algebra to confirm that



   (a) the **sum** $\alpha + \beta + \gamma$ of the three angles shown is $\pi/2$

   (b) the **real** part of any solution of $(z+1)^8 = (z-1)^8$ must be zero

2. For any real x and $N = 0,1,2,3,\ldots$, evaluate the sum

$$S_N(x) = \sum_{k=-N}^{N} e^{ikx}$$

as the **ratio of two sines**, and verify afterwards that this formula really works for both $N = 0$ and $N = 1$.

3. The inventor of the brilliant new function

$$w = \text{itch}(z) = 2e^z + e^{2z}$$

wants to know its **inverse** $\text{itch}^{-1}(w)$ explicitly in terms of the complex logarithm. Please help ... and also show the power of your formula (or of your logic) by reporting **all** possible values of z for which $\text{itch}(z) = 3$.

4. Evaluate $\displaystyle\oint_{|z|=1} z^4 e^{1/z}\, dz$. HINT: Think of **Laurent**

5. Evaluate $\displaystyle\int_0^\infty \frac{dx}{x^6+1}$. HINT: A slice of **pizza**

6. Show that the function $w(z) = z + \frac{1}{z}$ indeed maps any circle $|z| = \text{const}$ from the z-plane into an **ellipse** in the w-plane, with foci at $w = \pm 2$. (Sure, you may have forgotten exactly how the focus of an ellipse was defined ... but if need be, as here is, please rederive that as well.)

7. Solve the **Laplace equation** $\nabla^2 T = 0$ OUTSIDE the unit circle, given that

$$T(r=1,\theta) = \cos^6\theta$$

and also that $T(r,\theta)$ approaches the constant value $5/16$ at all large radii r.

HINT: Remember z^{-n}

8. Find the first three coefficients a_0, a_2 and a_4 needed in this **Fourier cosine series** of period π:

$$|\sin x| = a_0 + a_2 \cos 2x + a_4 \cos 4x + \ldots$$

PS: Since most of you are about to leave town, let's skip the usual funny business with code names for grade-postings outside an office door. Instead, a discreet email from you after this coming weekend will probably disclose your final grade very swiftly and reliably, even from afar! Now have a good summer ... AT + MB

(1) a) Consider $(3+i)(2+i)(1+i) = (5+5i)(1+i) \stackrel{!}{=} \boxed{10i}$, with $\measuredangle = 90°$ ✓

b) Here $|z+1| = |z-1|$, or $(x+1)^2+y^2 = (x-1)^2+y^2 \longrightarrow \boxed{4x=0}$ ✓

(2) Here $\boxed{S_N(x) = \sin\left(N+\tfrac{1}{2}\right)x \Big/ \sin\left(\tfrac{1}{2}x\right)}$, as explained most recently as the soln. to Prob (2) from Exam #1.

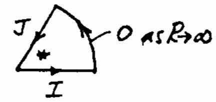
The special case $N=0$ yields $S_0(x)=1$ most reasonably, whereas for $N=1$ we expect $S_1(x) = e^{-ix}+1+e^{+ix} = 1+2\cos x$ and get $NUM = \sin\left(x+\frac{x}{2}\right) = \sin x\cos\frac{x}{2} + \cos x\sin\frac{x}{2}$ $= 2\sin\frac{x}{2}\cdot\cos^2\frac{x}{2} + \cos x\cdot\sin\frac{x}{2} = \sin\left(\tfrac{1}{2}x\right)\cdot\left\{(1+\cos x)+\cos x\right\} \longrightarrow S_1(x) = \text{same thing}$ ✓

(3) Here $z \stackrel{!}{=} \boxed{\log\left(\sqrt{w-1}-1\right) \equiv i\,\mathrm{tch}^{-1}(w)}$ and $w=3 \rightarrow z =$ __either__ $\boxed{2\pi i N}$

(For more detail, see soln. to Prob (3) from S97 Exam #1) __or else__ $\boxed{\ln 3+\pi i+2\pi i N}$

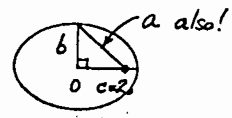
(4) Since $e^{1/z} = 1+\frac{1}{z}+\frac{1}{2!z^2}+\frac{1}{3!z^3}+\ldots$, the residue $Res(0)$ of $z^4e^{1/z}$ is the coefficient $\frac{1}{5!}$ of the resulting $\frac{1}{z}$ term. Hence $\oint z^4e^{1/z}dz \stackrel{!}{=} \boxed{2\pi i/5!}$

(5) The integral $I = \int_0^\infty \frac{dx}{x^6+1} \stackrel{!}{=} \boxed{\frac{\pi}{3}}$, as can be inferred from



and $J = \int_{r=\infty}^{0} \frac{e^{i\pi/3}\cdot dr}{r^6+1}$ and the residue $Res\left(e^{i\pi/6}\right) = \frac{e^{i\pi/6}}{-6}$.

(6) The circle $z = R\cos\theta + iR\sin\theta$ translates into $\boxed{\frac{u^2}{a^2}+\frac{v^2}{b^2}=1}$



$w = \left(R+\frac{1}{R}\right)\cos\theta + i\left(R-\frac{1}{R}\right)\sin\theta \equiv u+iv = a\cos\theta+ib\sin\theta$:

Hence indeed the squared focal distance $\boxed{c^2} \stackrel{!}{=} a^2-b^2 = \left(R+\frac{1}{R}\right)^2+\left(R-\frac{1}{R}\right)^2 \stackrel{!}{=} \boxed{4}$ ✓

(7) Since $\cos^6\theta = \frac{1}{2^6}\left(e^{i\theta}+e^{-i\theta}\right)^6 = \frac{1}{32}\cos 6\theta + \frac{6}{32}\cos 4\theta + \frac{15}{32}\cos 2\theta + \frac{20}{64}$, $\frac{20}{64} = \frac{5}{16}$,

our desired exterior solution $\boxed{T(r,\theta) = \frac{5}{16} + \frac{15\cos 2\theta}{32r^2} + \frac{3\cos 4\theta}{16r^4} + \frac{\cos 6\theta}{32r^6}}$

(8) Here $\boxed{|\sin x| = \frac{2}{\pi} - \frac{4}{3\pi}\cos 2x - \frac{4}{15\pi}\cos 4x - \ldots}$ to the extent that we requested

... although in fact this also equals $\frac{2}{\pi}\left\{1-\frac{2}{1\cdot 3}\cos 2x - \frac{2}{3\cdot 5}\cos 4x - \frac{2}{5\cdot 7}\cos 6x - \ldots\right\}$ from which, since $\frac{2}{1\cdot 3} = \frac{1}{1}-\frac{1}{3}$, $\frac{2}{3\cdot 5} = \frac{1}{3}-\frac{1}{5}$, $\frac{2}{5\cdot 7} = \frac{1}{5}-\frac{1}{7}$, etc, it is __FUN__ to sum this at $x=0$ to obtain $\frac{2}{\pi}\{1-1\} = 0$ very convincingly! And even at $x=\frac{\pi}{2}$ this checks nicely, since $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots = \frac{\pi}{4}$ = Wallis formula

Now have a NICE SUMMER!
AT

18.04 Final Exam from Fall 1997

Thursday, May 22, 1997 Time: 1:30 - 4:30 pm

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Final Examination in

18.04 Complex Variables with Applications

NOTE: Students are not allowed to use any books or notes during this examination. If brought into the room, such items must not be left nearby.

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1. If $|z| = 1$, prove that:

   a) $|z - w| = |1 - \bar{w}z|$ for any w

   b) $\mathrm{Re}[1/(1-z)] = 1/2$, excluding only $z = 1$

2. For the function $f(z) = \dfrac{\sin z \ \cos 3z}{z^4}$:

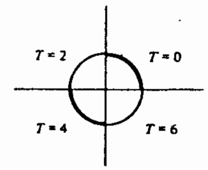
   a) Find the first three non-zero terms of its innermost Laurent expansion about $z = 0$

   b) Integrate $\oint_{|z|=1} f(z)\,dz$ counterclockwise once around $|z| = 1$

3. Use residue calculus to evaluate:

   a) $I = \displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2 + x + 1}$ b) $J = \displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^4 + x^2 + 1}$

4. Find the steady-state or "harmonic" temperature at the center of a unit disk if the temperature at its rim is:

   a) 

   ... and why? [2]

   b) $T(1,\theta) = \cos^4\theta$ [3]

   c) $T(1,\theta) = 3 / (2 - \cos\theta)$ [5 pts]

5. Figure out that Fourier series of period π (and also 2π) which closely imitates the non-negative function

   $$f(x) = |\cos x|$$

6. Regarding Möbius transformations like $w = \dfrac{az + b}{cz + d}$:

   a) Show that such transformations can never have more than two fixed points (at which $w(z) = z$) unless they are just trivial "carbon copies" with $w = z$ everywhere

   b) Locate both fixed points for $w = (z-i) / (z+i)$

   c) Find a different $w = (az+b) / (cz+d)$ that maps the points $z = 0, 1, \infty$ into $w = 0, 1, 2$, respectively

18.04 Brief Solutions to Final Exam Thu 22 May 97

(1) a) $|z-w| = |\overline{z-w}| = |\bar{z}-\bar{w}| = \left|\frac{1}{z}-\bar{w}\right| = \frac{|1-\bar{w}z|}{|z|} = |1-\bar{w}z|$ ✓

since $z\bar{z}=1$ here; likewise $=1$

b) Writing $z=e^{i\theta}$:

$$\frac{1}{1-z} = \frac{1}{1-e^{i\theta}} = \frac{1}{(1-\cos\theta)-i\sin\theta} = \frac{(1-\cos\theta)+i\sin\theta}{(1-\cos\theta)^2+\sin^2\theta}$$ ✓ since $2(1-\cos\theta) = \text{DEN}$

(2) a) $f(z) = \frac{1}{z^4}\left(z-\frac{z^3}{6}+\frac{z^5}{120}-\dots\right)\left(1-\frac{9z^2}{2}+\frac{81z^4}{24}-\dots\right)$

$\boxed{= \frac{1}{z^3} - \frac{14}{3}\frac{1}{z} + \frac{62}{15}z - \dots}$ since $\text{Res} = -\frac{14}{3}$: b) $\oint f\,dz \boxed{= -\frac{28\pi}{3}i}$

(3) a) $I \boxed{= 2\pi/\sqrt{3}}$ ...from simple pole at $z=e^{2\pi i/3}$, w/ $\text{Res}=\frac{1}{i\sqrt{3}}$

b) $J \boxed{= \pi/\sqrt{3}}$ ...from SP's at $z=e^{\pi i/3}, e^{2\pi i/3}$... $\sum \text{Res} = \frac{-i\sqrt{3}}{6}$

(4) a) $T(0,0) \boxed{= 3}$ ... literally the mean of those rim values

b) $T(0,0) = \langle \cos^4\theta \rangle \boxed{= \frac{3}{8}}$ c) $\dots = \frac{3}{2\pi}\int_{\theta=0}^{2\pi}\frac{d\theta}{2-\cos\theta} \boxed{= \sqrt{3}}$

(5) $f(x) = |\cos x| = a_0 + a_2\cos 2x + a_4\cos 4x + \dots$ yes, only the EVEN # cosines, by symmetry

and also $a_0 = \langle|\cos x|\rangle \boxed{= \frac{2}{\pi}}$.

The others are (for $k=1,2,3,\dots$) $a_{2k} \equiv \frac{1}{\pi}\int_{x=0}^{2\pi}|\cos x|\cos 2kx\,dx \boxed{= (-1)^{k+1}\frac{4/\pi}{4k^2-1}}$

$$= \frac{4}{\pi}\int_{x=0}^{\pi/2}\frac{e^{ix}+e^{-ix}}{2}\cdot\frac{e^{i2kx}+e^{-i2kx}}{2}\,dx = \frac{2}{\pi}\int_{x=0}^{\pi/2}[\cos(2k+1)x+\cos(2k-1)x]\,dx$$

(6) a) If $\frac{az_0+b}{cz_0+d} = z_0$, then of course $cz_0^2+(d-a)z_0-b=0$

with 2 solns at most, UNLESS $c=0$ AND $d=a$ AND $b=0$ too.

b) For $\frac{z_0-i}{z_0+i} = z_0$ in particular, $\boxed{z_0 = \frac{1\pm\sqrt{3}}{2}(1-i)}$

c) Need $b=0$, $c+d=a$, $d=c$. Thus $\boxed{w = \frac{2z}{z+1}}$ will do fine

PS: Only if you REALLY need to know your final grade in a hurry, send me an email by Friday morning.
I leave on a weeklong trip Friday afternoon. Good summer, etc.

AT