Tuesday, December 16, 2003 Time: 9 am - 12 noon

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Final Examination in

18.04 Complex Variables with Applications

NOTE: Students are not allowed to use any books or notes during this examination. If brought into the room, such items must not be left nearby.

Personal calculators may be used, but not shared.

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(1) Please explain again why $\frac{\pi}{4} = 4\tan^{-1}(1/5) - \tan^{-1}(1/239)$.

(2) Somehow or other, demonstrate again that $\frac{d}{dz}\tan^{-1} z = \frac{1}{1+z^2}$ even for complex z's .

(3) (a) Figure out the first five coefficients $a_o, a_1, \ldots, a_4$ needed in the expansion

$$\frac{1}{1+e^z} + \frac{A}{z^2+\pi^2} = a_o + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \ldots$$

(b) Other than craftily, how should the constant A have been chosen to permit such a Taylor series to converge even at $z = 2\pi i$?

(4) Let $f(z) = u(x,y) + i\,v(x,y)$ be an entire [= everywhere analytic] function of z , about which you are otherwise told only that on the real axis

$$u(x,0) = x\,, \quad v(x,0) = 2x \quad \text{for all } x\,.$$

Estimate both $f(2+i)$ and your likely error.

PROBLEMS 5,6,7,8 ⟶

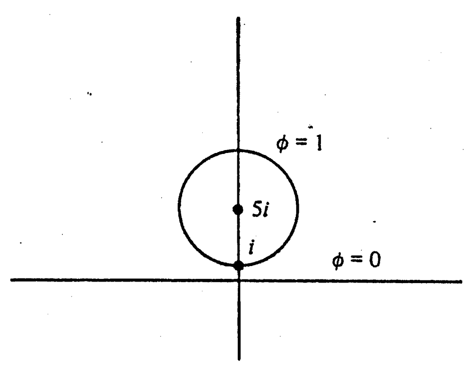
5. For real $0 < a < 1$, demonstrate anew that

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{x}}\,dx = \frac{\pi}{\sin a\pi} .$$

6. If C is the circle $|z| = 2$ described in the positive sense, evaluate the integral

$$\oint_C \tan z \, dz .$$

7. Probably via some bilinear transformation, find a function $\phi(x,y)$ which is harmonic in the portion of the upper half-plane exterior to the circle $|z - 5i| = 4$ and which takes the value $+1$ on that circle and the value 0 on the real axis — like in this figure copied from the book:



8. Find the Fourier representation for the <u>periodic</u> solutions $f(t)$ of the equation:

$$\frac{d^2f}{dt^2} + 2\frac{df}{dt} + f = \cos^2 t .$$