Exam Number 1 for 18.04, MIT (Fall 1999).

Due at the Lecture of Friday October 15, 1999.

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Contents

1 Points.

Problems 1.1, 1.2, 1.4 and 1.5 are worth 20 points each. Problems 1.3 and 1.6 are worth 10 points each. Problem 1.3 is too easy to be worth the same as the others, while problem 1.6 is a bit of a "challenge" problem, so do not be too concerned if you get stumped by it.

Warning: Present a solution **only** if you have some reasonably good idea of what is going on in a problem. Do not just write something in the hope of getting some partial credit; partial credit will be given when it is due, but **NEGATIVE CREDIT** will accrue for any gross error or similar (i.e.: beware of writting nonsense!) Please, read the instructions for more details on the rules that apply.

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2   Instructions.

1. **Penalties may/will result for failure to follow the rules below.**

2. **Do it alone.** You **can consult only** with the lecturer and/or the recitation instructors.

3. You can use the textbook, your class notes, the hand outs and the problem set answers. **Nothing else.** Be **specific** with the references (e.g.: "Using Theorem 87, p. 986 in the book") and **explain** how the reference fits into the answer. You **cannot** use the answers to the problems at the end of the book (except for those assigned, where a solution was provided with the problem set answers).

4. There is no time limit, but a few hours should be enough.

5. **Write the answers to each problem on separate pages, and your name and problem being solved at the head of each page** (e.g.: *18.04, J. Doe, Exam #3, Problem #77, page 2 of 29.* — meaning that your exam answer has 29 pages, this is page 2 and you are doing problem # 77).

6. **Staple** the whole exam.

7. In all your solutions show your reasoning, explaining carefully what you are doing. **Use English, not just mathematical symbols.** You play dice with unjustified steps (maybe I'll buy them, maybe not; if not: tough luck). This does not mean that you have to justify $2 + 2 = 4$, use some judgment here. Believe it or not, this is one skill you should learn at MIT: to write clearly your ideas. The place to learn this is in every course you take, not just when doing the writing requirements.

   Please: no "chicken scratches" or arrows on the side of the page leading from one piece of an argument to another and so on. *If a particular thing is illegible,* **write it again.** The **answers MUST be readable.** Put a box around your **final answer** (or use any other device you want, but make it so it **can be found easily).**

   Heavy penalties may be assessed for failure to follow this rule. **Do not** complain if you loose points this way. Arguments along the lines "what I meant to say ..." or "look, here is the answer" (on the margin, partly erased/crossed out, written in ambiguous way, etc) or similar will not work ... and may even lead to a "backwards" revision of the points.

8. Start **early.** Do not wait till the night before it is due.

9. The problems are actually not hard nor do they require much cumbersome calculation, **if** you think about them carefully and know what you are doing and where you are going. If you find yourself in the middle of a long messy answer, **stop** and think again about your strategy. It may not be a good way (maybe not even a way) to solve the problem.

   Another point: forget calculators. None of the problems require them, except to evaluate stuff like $\sqrt{5}$ in the final answer — not actually required nor even desired. If you start evaluating numerical

constants too early, you'll loose the "pattern" behind the problem and will increase your chances for a wrong answer (for example, you will miss some cancellation that makes something zero and knocks off a whole bunch of algebra out of the way). On the other hand, do not carry around things (like $\exp(i\pi)$) that you can evaluate exactly.

3   Problems.

3.1   Problem 1999.1.1.

Find all the branch points for the following complex valued multiple valued function and chose a set of branch cuts that make it single valued:
$$f(z) = \sqrt{\frac{z+1}{z-1}}.$$
Justify your answer. In particular, explain:

- Why the branch points you selected are the only ones.

- Show explicitly that the selected set of branch cuts lets you define a single valued analytic function in the "cut" complex plane.

Hint 3.1.1 *Define an appropriate set of angles (e.g.: θ_1 and θ_2) and radii (e.g.: r_1 and r_2). Then write an explicit formula for $f(z)$ — recall the approach used in the lectures for $\sqrt{z^2 - 1}$.*

3.2   Problem 1999.1.2.

Consider the multiple valued $\sec^{-1}(z)$ function in the complex plane, defined by:

$$\sec(w) = z \,,$$

where $w = \sec^{-1}(z)$. Find (and explain the steps leading to it) an expression for this function involving only logarithms, square roots and simple arithmetic operations. Use the expression obtained to find all the branch points points of the $\sec^{-1}(z)$ function.

3.3   Problem 1999.1.3.

What is the image, under the exponential map $w = e^z$, of the set of points in the Complex Plane defined by:

$$0 \le \mathrm{Re}(z) \le \ln(2) \quad \text{and} \quad 21\pi \le \mathrm{Im}(z) \le 23\pi \,,$$

where $\ln(x)$ denotes the real valued natural logarithm of a positive real number $x > 0$, as usual.

3.4   Problem 1999.1.4.

1. Let $\alpha > 0$ be some arbitrary real number. Show that $u = r^\alpha \sin(\alpha\theta)$ is an harmonic function (defined where?) and find it's harmonic conjugate v. What analytic function is $f(z) = u + iv$?

2. Use the results in part 1 to describe a two dimensional, irrotational, incompressible fluid flow in a wedge of arbitrary angle (i.e.: find a stream function that is constant — say, it vanishes — along the wedge walls). Sketch the flow lines in a few typical cases, in particular when the wedge angle is 2π.

In part 2 (specially) be careful with multiple valued functions, using appropriate branch cuts to make them single valued. When the wedge angle is 2π you will not have much choice as to where to put the branch cut; is there some physical interpretation you can give to this last flow and the role of the branch cut in the solution?

3.5   Problem 1999.1.5.

Consider the multiple valued function in the complex plane given by

$$f(z) = (\frac{8}{7}z^3 - \frac{64}{7})^{\frac{1}{3}}\,.$$

Show that the branch points of $w = f(z)$ are:

- $z_1 = 2$,

- $z_2 = 2\exp(+i\frac{2}{3}\pi)$,

- $z_3 = 2\exp(-i\frac{2}{3}\pi)$,

and nothing else (in particular ∞ is **not** a branch point — show this).

Consider now the branch of $f(z)$ (call it $F(z)$) defined (in the the complex plane minus two branch cuts) as follows:

- Branch cut **A**: straight line segment from z_1 to z_2.

- Branch cut **B**: straight line segment from z_1 to z_3.

- $\text{Im}(F(10)) = 0$.

What is the value of $F(1)$? Would this value change if we took branch cut **B** as the line segment from z_2 to z_3? What would the value be then?

3.6   Problem 1999.1.6.

Use Newton's method to find all the roots of

$$P(z) = 16z^5 - 20z^3 + 5z + 1 = 0\,,$$

with an accuracy: $|\text{error}| < 10^{-6}$ (note that the next part of the problem shows that all these roots are real numbers).

  By looking at the real part of the equation $(\exp(i\theta))^5 + 1 = 0$, show that the exact solutions of the problem above are:

- 1) $z_1 = \cos(\frac{1}{5}\pi)$ .... a double root,

- 2) $z_2 = \cos(\frac{3}{5}\pi)$ .... a double root,

- 3) $z_3 = \cos(\frac{5}{5}\pi) = -1$ .... a simple root.

That is, in fact one has $P(z) = (z - z_1)^2(z - z_2)^2(z - z_3)$. **Note:** you could probably **brute-force** your way into showing this last equality, but this is **not what is expected here.** With a bit of cleverness, you should be able to do this second part with a not-too-large amount of algebra.

THE END.