

Answers to Problem Set Number 9 for 18.04.

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1 Problems from the book by Saff and Snider.

1.1 Problem 06 in section 6.3.

We want to calculate the integral

$$I = \int_0^{\infty} \frac{x^2}{(x^2 + 1)(x^2 + 4)} dx,$$

which we can recast into the form

$$I = \frac{1}{2} \int_{-\infty}^{\infty} \frac{x^2}{(x^2 + 1)(x^2 + 4)} dx = \lim_{R \rightarrow \infty} \frac{1}{2} \int_{-R}^R \frac{x^2}{(x^2 + 1)(x^2 + 4)} dx,$$

where we have used that the integrand is an even function of x . We now turn the second integral (the one with the limit $R \rightarrow \infty$) into a contour integral in the complex plane, by closing the contour in the upper half plane with a semicircle of radius R . The contribution to the integral from the semicircle vanishes as $R \rightarrow \infty$, since the degree of the denominator is 2 higher than the degree of the numerator. Thus, let $f(z) = \frac{z^2}{(z+i)(z-i)(z+2i)(z-2i)}$. Then, using **residues**:

$$\begin{aligned} I &= \frac{2\pi i}{2} (\text{Res}(f, z=i) + \text{Res}(f, z=2i)) \\ &= \pi i \left(\frac{-1}{(2i)(3i)(-i)} + \frac{-4}{(3i)(i)(4i)} \right) \\ &= \frac{\pi}{6} \end{aligned}$$

1.2 Problem 11 in section 6.3.

Here we are interested in the integral

$$I = \int_0^{\infty} \frac{dx}{1+x^3}.$$

In order to do this integral, we consider the integral of $f(z) = 1/(1+z^3)$ over the path Γ given by the boundary of the wedge

$$S_R = \left\{ z = re^{i\theta} : 0 \leq \theta \leq \frac{2}{3}\pi \text{ and } 0 \leq r \leq R \right\}$$

and then let $R \rightarrow \infty$.

When $R \rightarrow \infty$ the contribution to the integral over Γ by the arc: $z = Re^{i\theta}$ ($0 \leq \theta \leq \frac{2}{3}\pi$) vanishes, since the degree of the denominator is 3 higher than the degree of the numerator. Thus only the

contributions from the two straight pieces in Γ remain. As $R \rightarrow \infty$, one of these pieces becomes the positive real axis, while the other can be parametrised by $z(x) = xe^{2\pi i/3}$, where $\infty > x \geq 0$ (notice that this is backwards in x , thus a minus sign will appear).

Since $f(xe^{2\pi i/3}) = f(x)$ and $f(z) = \frac{1}{(z - e^{i\pi/3})(z + 1)(z - e^{5\pi i/3})}$, we have

$$\lim_{R \rightarrow \infty} \int_{\Gamma} f(z) dz = \int_0^{\infty} f(x) dx + \int_0^{\infty} f(xe^{2\pi i/3}) e^{2\pi i/3} dx = (1 - e^{2\pi i/3}) \int_0^{\infty} f(x) dx,$$

and (using the residue theorem)

$$\lim_{R \rightarrow \infty} \int_{\Gamma} f(z) dz = 2\pi i \operatorname{Res}(f, z = e^{i\pi/3}) = \frac{2\pi i}{(e^{i\pi/3} + 1)(e^{i\pi/3} - e^{i5\pi/3})}.$$

Hence,

$$\begin{aligned} I = \int_0^{\infty} f(x) dx &= \frac{2\pi i}{(e^{i\pi/3} + 1)(e^{i\pi/3} - e^{i5\pi/3})(1 - e^{2\pi i/3})} \\ &= \frac{-2\pi i}{e^{i\pi/3} e^{i\pi/6} (e^{i\pi/6} + e^{-i\pi/6})(e^{i\pi/3} - e^{-i\pi/3})^2} \\ &= \frac{-2\pi}{2(2i)^2 \cos(\pi/6) \sin^2(\pi/3)} = \frac{\pi}{4(\sqrt{3}/2)(3/4)} \\ &= \frac{2\pi\sqrt{3}}{9}. \end{aligned}$$

1.3 Problem 01 in section 6.4.

We have

$$I = \int_{-\infty}^{\infty} \frac{\cos(2x)}{x^2 + 1} = \operatorname{Re} \left[\int_{-\infty}^{\infty} \frac{e^{2ix}}{(x+i)(x-i)} \right] = \lim_{R \rightarrow \infty} \operatorname{Re} \left[\int_{-R}^R \frac{e^{2ix}}{(x+i)(x-i)} \right].$$

We now close the contour (in the real line, from $-R$ to R) by a semicircle of radius R in the upper half plane. The contribution from this semicircle vanishes as $R \rightarrow \infty$ (by Jordan's lemma). Thus, from the residue theorem:

$$\begin{aligned} I &= \operatorname{Re} \left[2\pi i \operatorname{Res} \left(\frac{e^{2iz}}{(z+i)(z-i)}, z = i \right) \right] \\ &= \operatorname{Re} \left[2\pi i \frac{e^{-2}}{2i} \right] \\ &= \frac{\pi}{e^2}. \end{aligned}$$

Remark 1.3.1 *The book puts a “principal value” label in front of the integral $I = \int_{-\infty}^{\infty} \frac{\cos(2x)}{x^2 + 1}$. As far as I can tell, this is superfluous. This integral converges absolutely; there is no “principal value” involved:*

A principal value label only makes sense for integrals that do not converge absolutely, in which case more than one meaning can be given to the answer (thus one must clarify which one is being used). Principal value usually means that the integral is being given a meaning by taking a symmetric limit near the regions producing the failure of absolute convergence.

1.4 Problem 03 in section 6.4.

We have

$$I = \int_0^{\infty} \frac{\cos(x)}{(x^2 + 1)^2} dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\cos(x)}{(x^2 + 1)^2} dx = \lim_{R \rightarrow \infty} \frac{1}{2} \operatorname{Re} \left[\int_{-R}^R \frac{e^{ix}}{(x^2 + 1)^2} dx \right],$$

where we have used that the integrand is an even function of x . Now write $f(z) = \frac{e^{iz}}{(z+i)^2(z-i)^2}$ and close the contour of integration by a semicircle of radius R in the upper half plane. The contribution to the integral from the semicircle vanishes as $R \rightarrow \infty$, due to Jordan's Lemma. We then have

$$\begin{aligned} I &= \frac{1}{2} \operatorname{Re} [2\pi i \operatorname{Res}(f(z), z=i)] \\ &= \frac{1}{2} \operatorname{Re} \left[2\pi i \lim_{z \rightarrow i} \frac{d}{dz} \left(\frac{e^{iz}}{(z+i)^2} \right) \right] \\ &= \frac{1}{2} \operatorname{Re} \left[2\pi i \lim_{z \rightarrow i} \frac{(z+i)^2 i e^{iz} - e^{iz} 2(z+i)}{(z+i)^4} \right] \\ &= \frac{1}{2} \operatorname{Re}(2\pi i / (2ie)) \\ &= \frac{\pi}{2e} \end{aligned}$$

1.5 Problem 03 in section 6.5.

Consider now

$$I = \text{P.V.} \int_{-\infty}^{\infty} \frac{e^{ix}}{(x-1)(x-2)} dx = \lim_{\epsilon \rightarrow 0} \left(\int_{-\infty}^{1-\epsilon} + \int_{1+\epsilon}^{2-\epsilon} + \int_{2+\epsilon}^{\infty} \right) \frac{e^{ix}}{(x-1)(x-2)} dx.$$

Note that this integral only makes sense as a principal value (as indicated above) because of the two poles on the real axis. We can introduce now one more limit¹ and write:

$$I = \lim_{R \rightarrow \infty} \lim_{\epsilon \rightarrow 0} \left(\int_{-R}^{1-\epsilon} + \int_{1+\epsilon}^{2-\epsilon} + \int_{2+\epsilon}^R \right) \frac{e^{ix}}{(x-1)(x-2)} dx.$$

We now close the contour of integration using three semicircles in the upper half plane. One large (of radius R) centered at $z = 0$ and two small ones (of radius ϵ) centered at $z = 1$ and $z = 2$ respectively.

The resulting contour is similar to the one in figure 6.16 (section 6.5, page 278) of the book. Then:

- As $R \rightarrow \infty$ the contribution from the large semicircle vanishes (by Jordan's lemma).
- As $\epsilon \rightarrow 0$ each one of the small semicircles picks up a contribution of $-\pi i$ times the corresponding residue of $f(z) = \frac{e^{iz}}{(z-1)(z-2)}$ (by Lemma 4, section 6.3, page 275 in the book).
- Because f has no singularities inside the closed contour, its integral over it vanishes.

Putting all these facts together, we obtain:

$$\begin{aligned} I &= i\pi(\text{Res}(f, z=1) + \text{Res}(f, z=2)) \\ &= i\pi(-e^i + e^{2i}) \end{aligned}$$

1.6 Problem 10 in section 6.5.

We consider now the integral $\int_0^\infty \frac{\sin^2(x)}{x^2} dx$. Note that the integrand is an even function of x , thus we can write:

$$\begin{aligned} I &= \int_0^\infty \frac{\sin^2(x)}{x^2} dx = \frac{1}{2} \int_{-\infty}^\infty \frac{\sin^2(x)}{x^2} dx = \frac{1}{2} \int_{-\infty}^\infty \frac{\frac{1}{2}(1 - \cos(2x))}{x^2} dx \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{4} \left(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^\infty \right) \frac{1 - \cos(2x)}{x^2} dx \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{4} \text{Re} \left[\left(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^\infty \right) \frac{1 - e^{2ix}}{x^2} dx \right] = \lim_{R \rightarrow \infty} \lim_{\epsilon \rightarrow 0} \frac{1}{4} \text{Re} \left[\left(\int_{-R}^{-\epsilon} + \int_{\epsilon}^R \right) \frac{1 - e^{2ix}}{x^2} dx \right]. \end{aligned}$$

Remark 1.6.1 Notice that the introduction of the limit $\epsilon \rightarrow 0$ in the second line is necessary because (although $(1 - \cos(2x))/x^2$ is not singular at $x = 0$) once the integral is transformed to the real part of a complex integral, a singularity at $x = 0$ appears.

¹The integral defining I has no convergence problems at $x = \pm\infty$, this extra limit is for the purpose of being able to use residues.

We now close the contour in the upper half plane with a large semicircle (of radius R) centered at $z = 0$ and a small semicircle (of radius ϵ) also centered at $z = 0$. The resulting closed contour is very similar to the one shown in figure 6.15 (section 6.5, page 277) of the book. Now:

- As $R \rightarrow \infty$ the contribution from the large semicircle vanishes.
- As $\epsilon \rightarrow 0$ the contribution from the small semicircle picks up a contribution of $-\pi i$ times the residue at $z = 0$ of $f(z) = \frac{1 - e^{2iz}}{z^2}$ (by Lemma 4, section 6.3, page 275 in the book — note that f has a simple pole at $z = 0$).
- Because f has no singularities inside the closed contour, its integral over it vanishes.

Putting it all together, we have:

$$\begin{aligned} I &= \frac{1}{4}i\pi \operatorname{Res}(f(z), z = 0) \\ &= \frac{i\pi}{4} \lim_{z \rightarrow 0} \frac{d}{dz}(1 - e^{2iz}) \\ &= \frac{i\pi}{4} \lim_{z \rightarrow 0} (-2ie^{2iz}) \\ &= \frac{\pi}{2}. \end{aligned}$$

1.7 Problem 01 in section 6.6.

Here we consider the integral:

$$I = \int_0^\infty \frac{\sqrt{x}}{1+x^2} dx,$$

where we take the positive square root of x . Take now the branch of the square root function (with branch cut along the positive real axis) defined by

$$\sqrt{z} = \sqrt{r}e^{i\frac{1}{2}\theta},$$

for $z = re^{i\theta}$, with $0 \leq \theta \leq 2\pi$. Let now $f(z) = \frac{\sqrt{z}}{1+z^2}$ and consider the integral of f over the “key-hole” contour Γ given by²:

- Start on $x = \epsilon$ and follow the real line, remaining on the $\operatorname{Im}(z) > 0$ side, all the way up to $x = R$. That is, we have $z = x + i0$, with $\epsilon \leq x \leq R$.

²See figure 6.21 (section 6.6, page 284) in the book

- Continue with the circle $z = Re^{i\theta}$, $0 \leq \theta \leq 2\pi$.
- Continue now moving on the real line, remaining on the $\text{Im}(z) < 0$ side, all the way up from $x = R$ back to $x = \epsilon$. That is, we have $z = x - i0$, with $R \geq x \geq \epsilon$.
- Finish the contour with the circle $z = \epsilon e^{i\theta}$, $2\pi \geq \theta \geq 0$.

Now we have, as $R \rightarrow \infty$ and $\epsilon \rightarrow 0$:

- The contribution from the circle of radius R vanishes, since the degree of the denominator exceeds that of the numerator by $1.5 > 1$.
- The contribution from the circle of radius ϵ vanishes, since the integrand is bounded near $z = 0$ and the length of the contour vanishes.
- The contribution from the straight line piece on $\text{Im}(z) > 0$ has I as its limit.
- The contribution from the straight line piece on $\text{Im}(z) < 0$ has the limit:

$$\int_{\infty}^0 \frac{\sqrt{x}e^{2\pi i}}{1+x^2} dx = -e^{i\pi} \int_0^{\infty} \frac{\sqrt{x}}{1+x^2} dx = I.$$

- The contour encloses two singularities (simple poles) at $z = i = e^{i\pi/2}$ and $z = -i = e^{3i\pi/2}$. Thus, since $f(z) = \frac{\sqrt{z}}{(z+i)(z-i)}$, we have:

$$\begin{aligned} \int_{\Gamma} f(z) dz &= 2\pi i (\text{Res}(f, z=i) + \text{Res}(f, z=-i)) \\ &= 2\pi i \left(\frac{e^{i\pi/4}}{2i} + \frac{e^{3i\pi/4}}{-2i} \right) \\ &= 2\pi i \left(\frac{e^{i\pi/4}}{2i} + \frac{e^{-i\pi/4}}{2i} \right) \\ &= 2\pi \cos \frac{\pi}{4} \\ &= 2 \frac{\pi}{\sqrt{2}}. \end{aligned}$$

Putting all this together, we get:

$$I = \frac{\pi}{\sqrt{2}}.$$

1.8 Problem 04 in section 6.6.

Here we consider the integral:

$$I = \int_0^\alpha \frac{x^\alpha}{(1+x^2)^2},$$

where $-1 < \alpha < 3$, $\alpha \neq 1$ and we take the real positive value for x^α . We proceed as in the solution to problem 01 in this section and define a branch for z^α (with the branch cut along the positive real axis) by

$$z^\alpha = r^\alpha e^{i\alpha\theta},$$

for $z = re^{i\theta}$, with $0 \leq \theta \leq 2\pi$. Then we let $f(z) = \frac{z^\alpha}{(1+z^2)^2}$ and consider the integral of f over the same “key-hole” contour Γ used in problem 01.

Again, as $R \rightarrow \infty$ and $\epsilon \rightarrow 0$ the only contributions that survive are those from the straight line pieces in Γ . Thus we have:

$$\begin{aligned} \lim_{R \rightarrow \infty} \lim_{\epsilon \rightarrow 0} \int_\Gamma \frac{z^\alpha}{(z^2+1)^2} dz &= \int_0^\infty \frac{x^\alpha}{(x^2+1)^2} dx + \int_\infty^0 \frac{(xe^{2\pi i})^\alpha}{(x^2+1)^2} dx \\ &= (1 - e^{2\pi i\alpha}) \int_0^\infty \frac{x^\alpha}{(x^2+1)^2} dx \\ &= (1 - e^{2\pi i\alpha}) I. \end{aligned}$$

On the other hand, the contour encloses singularities (poles) at $z = \pm i$. Thus, from the Residue theorem:

$$\begin{aligned} \int_0^\infty f(x) dx &= \frac{2\pi i}{1 - e^{2\pi i\alpha}} (\text{Res}(f, z=i) + \text{Res}(f, z=-i)) \\ &= \frac{2\pi i}{1 - e^{2\pi i\alpha}} \left[\lim_{z \rightarrow i} \frac{d}{dz} \left(\frac{z^\alpha}{(z+i)^2} \right) + \lim_{z \rightarrow -i} \frac{d}{dz} \left(\frac{z^\alpha}{(z-i)^2} \right) \right] \\ &= \frac{2\pi i}{1 - e^{2\pi i\alpha}} \left[\lim_{z \rightarrow i} \frac{(z+i)\alpha z^{\alpha-1} - 2z^\alpha}{(z+i)^3} + \lim_{z \rightarrow -i} \frac{(z-i)\alpha z^{\alpha-1} - 2z^\alpha}{(z-i)^3} \right] \\ &= \frac{\pi}{2(1 - e^{2\pi i\alpha})} \left[-\alpha e^{i\pi\alpha/2} + e^{i\pi\alpha/2} + \alpha e^{3i\pi\alpha/2} - e^{3i\pi\alpha/2} \right] \\ &= \frac{\pi}{2} (1 - \alpha) \frac{e^{-i\pi\alpha/2} - e^{i\pi\alpha/2}}{e^{-i\pi\alpha} - e^{i\pi\alpha}} \\ &= \frac{\pi}{2} (1 - \alpha) \frac{\sin(\pi\alpha/2)}{\sin(\pi\alpha)} \\ &= \frac{\pi}{4} \frac{1 - \alpha}{\cos(\pi\alpha/2)}. \end{aligned}$$

1.9 Problem 06 in section 6.7.

We want to show that the polynomial $p(z) = z^6 + 4z^2 - 1$ has exactly two roots inside the circle $|z| = 1$. To this end, we use Rouché's Theorem (Theorem 4, section 6.7, page 293 in the book). Let $f(z) = 4z^2 - 1$ and $h(z) = z^6$. Then, on $|z| = 1$

$$|f(z)| \geq |4|z^2| - 1| = 3 > 1 = |z^6| = |h(z)|.$$

Since f and h are clearly analytic for $|z| < 1$, the hypothesis of the theorem are satisfied. Hence $p = f + h$ and f have the same number of roots in $|z| < 1$. Since the roots of f are $z = \pm \frac{1}{2}$, this proves the result.

Note: it need not be that p has two distinct roots in $|z| < 1$; it could have a single (double) root.

We note that the split above is not unique. We could take $f(z) = z^6 + 4z^2 = z^2(z^4 + 4)$ and $h(z) = 1$. This works too, because f has a zero of order 2 at the origin and 4 more zeros, all of which satisfy $|z| = \sqrt{2}$ (so they lie outside the unit disc). Furthermore: on the unit circle $|z| = 1$, we have:

$$|f(z)| = |z^6 + 4z^2| = |z^2||z^4 + 4| = |z^4 + 4| \geq ||z|^4 - 4| = 3 > 1 = |h(z)|.$$

1.10 Problem 10 in section 6.7.

We want to show that the equation $p(z) = 2 - z - e^{-z} = 0$ has one root in the right half plane. Consider the functions $f(z) = 2 - z$ and $h(z) = -e^{-z}$. The function $f(z)$ clearly has one root in the right half plane (at $z = 2$). Furthermore, on the imaginary axis (write $z = iy$, with $-\infty < y < \infty$) we have:

$$|f(z)| = |2 - iy| = \sqrt{2^2 + y^2} \geq 2 > 1 = |-e^{-iy}| = |h(z)|.$$

Thus $f(z)$ and $p = f + h$ must have the same number of zeros in the right half plane (namely 1), by Rouché's Theorem.

Note that if z_0 is a root of $z = 2 - e^{-z}$, then \bar{z}_0 must also be a root of the equation. But if z_0 is in the right half plane, then so is \bar{z}_0 . Since we know that there is only one root in this region, we conclude that $z_0 = \bar{z}_0$, i.e.: **the root is real.**

1.11 Problem 04 in section 7.1.

We consider here the problem of finding a **function Φ that is harmonic in the right hand plane** and takes the values

$$\Phi(0, y) = \frac{y}{1 + y^2} \quad (1.11.1)$$

on the imaginary axes. The obvious guess: $\Phi = \text{Im}\left(\frac{z}{1 - z^2}\right)$, does not work because $\frac{z}{1 - z^2}$ is not analytic at $z = 1$. A subtler strategy is called for.

Consider the correspondence between the right half plane in the z -plane and the unit disk in the w -plane given by the map from z to w (and its inverse):

$$w(z) = \frac{z - 1}{z + 1} \quad \text{and} \quad z(w) = \frac{1 + w}{1 - w}. \quad (1.11.2)$$

Then $\Psi(w) = \Phi(z(w))$ defines a function that is harmonic on the unit disk $|w| < 1$, with values on the boundary $w = e^{i\theta}$ which we can calculate via the correspondence (1.11.2) and (1.11.1). Let us now calculate these values.

(a) The correspondence between the unit disk and right hand plane boundaries is given by

$$z(e^{i\theta}) = \frac{1 + e^{i\theta}}{1 - e^{i\theta}} = \frac{(1 + e^{i\theta})(1 - e^{-i\theta})}{(1 - e^{i\theta})(1 - e^{-i\theta})} = i \frac{\sin \theta}{1 - \cos \theta} = i y(\theta).$$

Thus, we have

$$\Psi(e^{i\theta}) = \Phi(0, y(\theta)) = \Phi\left(0, \frac{\sin \theta}{1 - \cos \theta}\right) = \frac{\sin \theta}{1 - \cos \theta} \cdot \frac{1}{1 + (\sin^2 \theta / (1 - \cos \theta)^2)} = \frac{\sin \theta}{2}.$$

(b) Since $f(w) = \frac{1}{2}w$ is an entire function, $\text{Im}\left(\frac{1}{2}w\right)$ is harmonic. Furthermore: $\text{Im}\left(\frac{1}{2}e^{i\theta}\right) = \frac{1}{2}\sin \theta$.

Thus we have an harmonic function that satisfies the correct boundary conditions. Since the solution of Laplace's equation with Dirichlet boundary conditions is unique, it follows that this must be the one and only correct solution. That is to say, it must be:

$$\Psi = \text{Im}\left(\frac{1}{2}w\right). \quad (1.11.3)$$

(c) To find Φ , now we take Ψ and use the mappings (1.11.2). That is:

$$\Phi(x, y) = \Psi(w(z)) = \text{Im}\left(\frac{1}{2}w(z)\right) = \text{Im}\left(\frac{z - 1}{2(z + 1)}\right) = \frac{1}{2}\text{Im}\left(\frac{x + iy - 1}{x + iy + 1}\right) = \frac{y}{y^2 + (x + 1)^2}.$$

1.12 Problem 07 in section 7.2.

Consider the mapping $w(z) = z + \frac{1}{z}$. An element of the circle $|z| = \rho$, say $z = \rho e^{i\theta}$, gets mapped to

$$w(\rho e^{i\theta}) = \rho e^{i\theta} + \frac{1}{\rho} e^{-i\theta} = \left(\rho + \frac{1}{\rho}\right) \cos \theta + i \left(\rho + \frac{1}{\rho}\right) \sin \theta.$$

So we have $u = \operatorname{Re}(w) = \left(\rho + \frac{1}{\rho}\right) \cos \theta$ and $v = \operatorname{Im}(w) = \left(\rho - \frac{1}{\rho}\right) \sin \theta$. Thus:

$$\frac{u^2}{\left(\rho + \frac{1}{\rho}\right)^2} + \frac{v^2}{\left(\rho - \frac{1}{\rho}\right)^2} = \cos^2 \theta + \sin^2 \theta = 1.$$

1.13 Problem 13 in section 7.2.

Here we consider the images of various sets in the complex plane by the map

$$w = \cos(z) = \cos(x) \cosh(y) - i \sin(x) \sinh(y),$$

where³ $z = x + iy$ and $w = u + iv$. Thus $u = \cos(x) \cosh(y)$ and $v = -\sin(x) \sinh(y)$.

Remark 1.13.1 *Analytic maps preserve orientation. Thus, when mapping boundaries of regions, it is useful to follow the boundary keeping the region (say) on the right. Then the image of the region will be on the right of the image of the boundary.*

We follow this strategy below: the sense in which any given curve is traversed is indicated by the order of the inequalities. Thus, if we write a line as $x = 0$ and $-\infty < y \leq 0$, we mean that the line is to be followed in the direction of increasing y . On the other hand $x = 0$ and $0 \geq y > -\infty$ means that the line is to be followed in the direction of decreasing y . The same strategy is used for the image curves.

To better understand the arguments that follow below, you should draw the various sets and put arrows on their boundaries (pointing in the direction such that the set is to the right). Then do the same for the images of the boundaries.

(a) **Half vertical strip Ω_a : $0 < x < \pi$ and $y < 0$.**

Lets look first at what happens on the boundary:

³That is: $x = \operatorname{Re}(z)$, $y = \operatorname{Im}(z)$, $u = \operatorname{Re}(w)$ and $v = \operatorname{Im}(w)$.

- Left side: $x = 0$ and $-\infty < y \leq 0$.
Mapped to the half real axis: $\infty > u \geq 1$ and $v = 0$.
This because $w = \cosh(y)$ and $-\infty < y \leq 0$.
- Top side: $y = 0$ and $0 \leq x \leq \pi$.
Mapped to the real interval: $1 \geq u \geq -1$ and $v = 0$.
This because $w = \cos(x)$ and $0 \leq x \leq \pi$.
- Right side: $x = \pi$ and $0 \geq y > -\infty$.
Mapped to the half real axis: $-1 \geq u > -\infty$ and $v = 0$.
This because $w = -\cosh(y)$ and $0 \geq y > -\infty$.

Thus we see that the boundary of Ω_a gets mapped to the entire real axis. Since the image of Ω_a must be to the right of the image of the boundary (when the boundary is traversed keeping Ω_a to the right), we conclude that:

The image of Ω_a is the upper half plane.

As a further check: consider a point $z_0 \in \Omega_a$. Then $\sin(x_0) > 0$ and $\sinh(y_0) < 0$. It follows that: $v_0 = -\sin(x_0)\sinh(y_0) > 0$.

(b) **Half vertical strip Ω_b : $0 < x < \frac{1}{2}\pi$ and $y > 0$.**

Lets look first at what happens on the boundary:

- Right side: $x = \frac{1}{2}\pi$ and $\infty > y \geq 0$.
Mapped to the half imaginary axis: $-\infty < v \leq 0$ and $u = 0$.
This because $w = -i\sinh(y)$ and $\infty > y \geq 0$.
- Bottom side: $y = 0$ and $\frac{1}{2}\pi \geq x \geq 0$.
Mapped to the real interval: $0 \leq u \leq 1$ and $v = 0$.
This because $w = \cos(x)$ and $\frac{1}{2}\pi \geq x \geq 0$.
- Left side: $x = 0$ and $0 \leq y < \infty$.
Mapped to the half real axis: $1 \leq u < \infty$ and $v = 0$.
This because $w = \cosh(y)$ and $0 \leq y < \infty$.

Thus we see that the boundary of Ω_b gets mapped to the lower half of the imaginary axis, followed by the positive real axis. Since the image of Ω_b must be to the right of the image of the boundary (when the boundary is traversed keeping Ω_b to the right), we conclude that:

The image of Ω_b is the fourth quadrant in the complex plane.

As a further check: consider a point $z_0 \in \Omega_b$. Then $\sin(x_0) > 0$, $\cos(x_0) > 0$, $\sinh(y_0) > 0$ and $\cosh(y_0) > 0$. It follows that: $u_0 = \cos(x_0) \cosh(y_0) > 0$ and $v_0 = -\sin(x_0) \sinh(y_0) < 0$.

(c) **Vertical strip Ω_c : $0 < x < \pi$.**

Lets look first at what happens on the boundary:

- Left side: $x = 0$ and $-\infty < y < \infty$ (i.e.: the imaginary axis).

Mapped to the half real axis: $u \geq 1$ and $v = 0$.

This because $w = \cosh(y)$ and $-\infty < y < \infty$. However, **watch out here:** notice that **the image is traversed twice**. As y goes from $-\infty$ to ∞ , u goes from ∞ to 1 and then back to ∞ (think of the implication of this, given that the image set is to the right of the image boundary).

- Right side: $x = \pi$ and $\infty > y > -\infty$.

Mapped to the half real axis: $u \leq -1$ and $v = 0$.

This because $w = -\cosh(y)$ and $\infty > y > -\infty$. Again: **the image is traversed twice**. As y goes from ∞ to $-\infty$, u goes from $-\infty$ to -1 and then back to $-\infty$.

Thus we see that the boundary of Ω_c gets mapped to the real axis, except for the segment $-1 \leq u \leq 1$. Each of the two pieces corresponds to one of the sides of Ω_c and each is covered twice as the boundary of Ω_c is followed (in the fashion explained above). Since the image of Ω_c must be to the right of the image of the boundary (when the boundary is traversed keeping Ω_c to the right), we conclude that:

The image of Ω_c is the entire complex plane, except for the segments on the real axis given by $-\infty < u \leq -1$ and $1 \leq u < \infty$.

For example: the point $z = \frac{1}{2}\pi \in \Omega_c$ is mapped to $w = 0$.

(d) **Rectangle Ω_d : $0 < x < \pi$ and $-1 < y < 1$.**

Lets look first at what happens on the boundary:

- Left side: $x = 0$ and $-1 \leq y \leq 1$.

Mapped to the real interval: $1 \leq u \leq \cosh(1)$ and $v = 0$.

This because $w = \cosh(y)$ and $-1 \leq y \leq 1$. But, **watch out, the image is traversed twice.** As y goes from -1 to 1 , u goes from $\cosh(1)$ to 1 and then back to $\cosh(1)$.

- Top side: $0 \leq x \leq \pi$ and $y = 1$.

Mapped to the lower half of the ellipse centered at the origin, with semi-axes $\cosh(1)$ in the real direction and $\sinh(1)$ in the imaginary direction. The motion along it is clockwise.

This because $w = a \cos(x) - ib \sin(x)$, where $a = \cosh(1)$, $b = \sinh(1)$ and $0 \leq x \leq \pi$. Thus $u = a \cos(x)$ and $v = -b \sin(x)$, which is a parametric form of an ellipse.⁴

- Right side: $x = \pi$ and $1 \geq y \geq -1$.

Mapped to the real interval: $-\cosh(1) \leq u \leq -1$ and $v = 0$.

This because $w = -\cosh(y)$ and $-1 \leq y \leq 1$. Again, **the image is traversed twice.** As y goes from 1 to -1 , u goes from $-\cosh(1)$ to -1 and then back to $-\cosh(1)$.

- Bottom side: $\pi \geq x \geq 0$ and $y = -1$.

Mapped to the upper half of the ellipse centered at the origin, with semi-axes $\cosh(1)$ in the real direction and $\sinh(1)$ in the imaginary direction. The motion along it is clockwise.

This because $w = a \cos(x) + ib \sin(x)$, where $a = \cosh(1)$, $b = \sinh(1)$ and $\pi \geq x \geq 0$. Thus $u = a \cos(x)$ and $v = b \sin(x)$, which is a parametric form of an ellipse.⁵

From the above, in the same fashion as in the other cases, we conclude that:

The image of Ω_d is the elliptical set $\left(\frac{u}{\cosh(1)}\right)^2 + \left(\frac{v}{\sinh(1)}\right)^2 < 1$, except for the segments on the real axis given by $v = 0$ and $-\cosh(1) < u \leq -1$ or $1 \leq u < \cosh(1)$.

⁴Note that $(u/a)^2 + (v/b)^2 = 1$. Also, $v \leq 0$, since $0 \leq x \leq \pi$.

⁵Note that $(u/a)^2 + (v/b)^2 = 1$. Also, $v \geq 0$, since $0 \leq x \leq \pi$.

For example: the point $z = \frac{1}{2}\pi \in \Omega_c$ is mapped to the origin.

1.14 Problem 05 in section 7.3.

The general Möbius transformation takes the form

$$w = u + iv = \frac{az + b}{cz + d},$$

where a, b, c and d are (complex) constants. In this problem we want to find a Möbius transformation that maps the unit disk $|z| < 1$ onto the right hand plane $u > 0$ and takes $z = -i$ to $w = 0$.

- The condition $w(-i) = 0$ means that we must have $a(-i) + b = 0$, that is $b = ia$.
- One of the points in the unit circle must go to the point at infinity (the unit disk must be mapped to the boundary of the right hand plane; that is: the imaginary axis). We are free to choose this point (the only point we cannot choose is $z = -i$). Thus select $z = 1$ for this purpose. Then we need $c \cdot 1 + d = 0$, that is $d = -c$.

Thus far we have

$$w = \frac{a}{c} \cdot \frac{z + i}{z - 1}.$$

Now, we need to make sure that all the points on the unit circle are mapped to the imaginary axis. Since Möbius transformations take circles into circles,⁶ we only need to make sure three points on the unit circle get mapped to the imaginary axis. We already have two; to get the third let us make sure that $z = -1$ maps to a point on the imaginary axis. Thus, we must have:

$$w(-1) = \frac{a}{c} \left(\frac{-1 + i}{-2} \right) = \frac{a}{\sqrt{2}c} \left(\frac{1}{\sqrt{2}} - i \frac{1}{\sqrt{2}} \right) = \frac{a}{\sqrt{2}c} e^{-i\pi/4} = i\alpha,$$

for some real constant α . So, choose $\frac{a}{c} = e^{i3\pi/4}$ (which corresponds to $\alpha = 1/\sqrt{2}$), to obtain

$$w = e^{(i3\pi/4)} \frac{z + i}{z - 1}.$$

This maps the unit circle onto the imaginary axis, but it need not map the inside of the unit circle to the right hand plane (it could map it to the left hand plane). However, $w(0) = e^{i\pi/4}$ which is indeed in the right half plane, and so we have a correct map.

⁶Straight lines are the special case of a circle through infinity.

Remark 1.14.1 *Notice that there are two basic ways of figuring out what the image of a set (by an analytic function) is. These are:*

- *See where the boundary of the set gets mapped, including orientation (say, go around the set keeping the set on the right and see which way the mapped curve is traversed). Since analytic maps preserve orientation, the image set will be on the same side of the image boundary as the set was relative to its boundary.*
- *Map the boundary without concern for orientation and then check where an arbitrary point inside the set gets mapped to. This will tell you which side of the image of the boundary the image of the set is on. This is more efficient (but less intuitive).*

THE END.