

Answers to Problem Set Number 6 for 18.04.

MIT (Fall 1999)

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1 Problems from the book by Saff and Snider.

1.1 Problem 04 in section 4.7.

When thermal equilibrium is reached in a bounded closed region, the steady state temperature can be described by a real-valued function $T = T(x, y)$, which is a solution of Laplace's equation

$$0 = \Delta T(x, y) = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2},$$

Thus T is a harmonic function (see Section 2.6 of the book for more details). A maximum of T corresponds to the “hottest spot” and a minimum corresponds to the “coldest spot”. The maximum-minimum principle for harmonic functions says that the maximum and minimum can not occur in the interior, which makes perfect sense in this physical context: the hottest spot cannot occur in the interior at equilibrium because heat would flow away from it to the surrounding area and it would cool down; similarly, the coldest spot cannot occur in the interior either since heat would flow into it from the surrounding area to warm it up.

1.2 Problem 05 in section 4.7.

Here we will find an unbounded domain and two harmonic functions, which agree on the boundary but are not identical in the domain. They will then be two different solutions to the same Dirichlet problem. This will show that **the solution to the Dirichlet problem on an unbounded domain need not be unique (unless extra¹ conditions on the problem are imposed)**.

Let us choose as the domain the right-half plane: $\text{Re}(z) = x > 0$, whose boundary is the imaginary axis: $\text{Re}(z) = x = 0$. Let us now consider the Dirichlet problem on this domain with vanishing values for the solution (a harmonic function) on the boundary. That, is, the problem is:

$$\Delta \Phi = 0 \text{ on } x > 0, \quad \text{with } \Phi(0, y) = 0 \text{ for all } y.$$

An obvious solution to this problem is the zero function $\Phi_1(x, y) = 0$. If the solutions to the Dirichlet problem were unique in this case, all the harmonic functions vanishing on the y -axis would vanish on the whole right-half plane. This is certainly not the case. Some simple counter-examples are

¹Typically, Dirichlet problems on infinite domains arising in applications will have such extra restrictions.

- $\Phi_2(x, y) = \sin(x) \exp(-y)$ (harmonic, since it is the real part of $\exp(iz)$).
- $\Phi_3(x, y) = 2xy$ (harmonic, since it is the imaginary part of z^2).

Notice that, if we require the solution to be bounded, these counter-examples are excluded.

1.3 Problem 11 in section 4.7.

To do this problem we use the *Mean Value Property for harmonic functions* (done in the lectures). The temperature at the center of the disk must be the average of the temperature on the rim. Since on the rim $T \equiv 0$, $T \equiv 6$, $T \equiv 4$ and $T \equiv 2$ on each of four equal sectors (spanning a $\frac{1}{2}\pi$ angle each), it follows that

$$T(0, 0) = 3.$$

1.4 Problem 2b in section 5.1.

We use the ratio test to check the convergence of the series

$$\sum_{k=1}^{\infty} c_k = \sum_{k=1}^{\infty} \frac{(3+i)^k}{k!}.$$

We have that

$$\lim_{k \rightarrow \infty} \left| \frac{c_{k+1}}{c_k} \right| = \lim_{k \rightarrow \infty} \frac{|3+i|}{k+1} = \lim_{k \rightarrow \infty} \frac{\sqrt{10}}{k+1} = 0 < 1.$$

Hence the series converges.

1.5 Problem 07 in section 5.1.

Here we will examine several series and determine if they converge or diverge. *We will use two tests for convergence.* The **first test** is sometimes called the *n-th term test* and it is a necessary condition for convergence. Namely:

If an infinite sum $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \rightarrow 0$ as $n \rightarrow \infty$.

Notice that, if this test is “passed”, the series may still diverge (as the harmonic series, with $a_n = \frac{1}{n}$, shows). It is only when $|a_n| \not\rightarrow 0$ as $n \rightarrow \infty$ that this test provides a definite answer.

The **second test is the ratio test**. The ratio test gives a definite answer when the ratio is strictly less than 1 (convergence) or strictly greater than 1 (divergence), but provides no information when the ratio is 1. In this last case one must use other tests, such as (for example) the first test above. Notice that, if the ratio test gives 1 for the ratio and the first test “passes”, the issue of convergence will remain undecided.

a) Series with: $a_n = \left(\frac{1+2i}{1-i}\right)^n$. The ratio test yields:

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{1+2i}{1-i} \right| = \sqrt{\frac{5}{2}} > 1.$$

Therefore this series diverges. The n-th term test also fails, since

$$|a_n| = \left(\sqrt{\frac{5}{2}}\right)^n \rightarrow \infty$$

as $n \rightarrow \infty$.

b) Series with: $a_n = \frac{1}{n^2 3^n}$. The ratio test yields:

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{1}{3} \left(\frac{n}{n+1}\right)^2 = \frac{1}{3} < 1.$$

Therefore this series converges.

c) Series with: $a_n = \frac{ni^n}{2n+1}$. We use the n-th term test:

$$\lim_{n \rightarrow \infty} |a_n| = \lim_{n \rightarrow \infty} \frac{n}{2n+1} = \frac{1}{2} \neq 0.$$

Therefore the series diverges. Notice that in this case the ratio test is not useful, since:

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{(2n+1)(n+1)}{(2n+3)n} = 1.$$

d) Series with: $a_n = \frac{n!}{5^n}$. The ratio test yields:

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{n+1}{5} = \infty.$$

Therefore the series diverges.

e) Series with: $a_n = \frac{(-1)^n n^3}{(1+i)^n}$. The ratio test yields:

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{1}{|1+i|} \left(\frac{n+1}{n} \right)^3 = \frac{1}{\sqrt{2}} < 1.$$

Therefore the series converges.

f) Series $a_n = i^n - \frac{1}{n^2}$. We use the n-th term test:

$$\lim_{n \rightarrow \infty} |a_n| = 1 > 0.$$

Therefore the series diverges. Again, in this case the ratio test is not useful (as it yields a ratio of one).

1.6 Problem 13 in section 5.1.

Here we show that the series $\sum_{j=1}^{\infty} j^{-p}$ converges for $p > 1$.

The sequence $\frac{1}{j^p}$ is non-negative, thus to show that its sum is convergent it is enough to show that it is bounded. Consider then the integral:

$$A_N = \int_1^N \frac{1}{x^p} dx = \sum_{j=1}^{N-1} \int_j^{j+1} \frac{1}{x^p} dx.$$

The integrand $\frac{1}{x^p}$ is decreasing when $p > 0$, so that

$$\frac{1}{x^p} > \frac{1}{(j+1)^p}$$

for $j \leq x \leq j+1$. Applying this inequality to the integral in each unit interval $[j, j+1]$ above, we get:

$$A_N = \sum_{j=1}^{N-1} \int_j^{j+1} \frac{1}{x^p} dx > \sum_{j=2}^N \frac{1}{j^p}.$$

On the other hand, A_N can be computed directly by integration:

$$A_N = \frac{1-p}{x^{p-1}} \Big|_1^N = (p-1) \left(1 - \frac{1}{N^{p-1}} \right) < p-1.$$

Therefore,

$$\sum_{j=1}^N \frac{1}{j^p} < 1 + A_N < p \quad \text{for all } N.$$

This shows that the series is convergent.

1.7 Problem 02 in section 5.2.

For an analytic function $f = f(z)$, the Taylor series centered at a point z_0 converges to $f(z)$ everywhere inside the largest open disk (centered at z_0) over which f is analytic. Thus to determine where the various Taylor expansions below converge, we just need to determine the analytic domain of the function f .

- a) $f(z) = e^{-z}$ is entire, thus its Taylor expansion (around any point) is valid over the whole plane.
- b) $f(z) = \cosh(z) = \frac{e^z + e^{-z}}{2}$ is entire, thus its Taylor expansion (around any point) is valid over the whole plane.
- c) $f(z) = \sinh(z) = \frac{e^z - e^{-z}}{2}$ is entire, thus its Taylor expansion (around any point) is valid over the whole plane.
- d) $f(z) = \frac{1}{1-z}$ is analytic everywhere, except at $z = 1$. Thus its Taylor series centered at $z_0 = i$ converges on the disk $|z - i| < |1 - i| = \sqrt{2}$.
- e) $f(z) = \text{Log}(1 - z)$ is analytic everywhere, except on $z = \text{Re } z \geq 1$. Therefore its Taylor series centered at $z_0 = 0$ converges on the disk $|z| < 1$.
- f) $f(z) = z^3$ is entire, thus its Taylor expansion (around any point) is valid over the whole plane.

1.8 Problem 04 in section 5.2.

Here we show that the following expansion

$$(1+z)^\alpha = e^{\alpha \text{Log}(1+z)} = \sum_{n=0}^{\infty} \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!} z^n, \quad (1.8.1)$$

is valid for $|z| < 1$ and any α (a complex number).

Equation (1.8.1) is just the Taylor expansion (around $z = 0$) for the function $f(z) = e^{\alpha \text{Log}(1+z)}$ (we show this below). To find its radius of convergence, we need to determine the largest disk (centered at $z = 0$) over which f is analytic. Since $\text{Log}(1+z)$ is analytic everywhere (except on the half of the real axis given by $x = \text{Re}(z) \leq -1$) and $e^{\alpha z}$ is entire, it follows that f (as their composition)

is analytic in the domain made up by the whole complex plane, with the half line $x = \operatorname{Re}(z) \leq -1$ excluded. Thus the Taylor expansion around $z = 0$

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n, \quad (1.8.2)$$

converges for $|z| < 1$.

Now we compute $f^{(n)}(0)$. It is easy to see that

$$\frac{df}{dz} = \alpha(1+z)^{(\alpha-1)},$$

so that the derivative of f has the same form as f itself (with $\alpha \rightarrow (\alpha - 1)$). Using induction it then follows that

$$\frac{d^n f}{dz^n} = \alpha(\alpha - 1) \dots (\alpha - n + 1)(1+z)^{\alpha-n},$$

so that

$$f^{(n)}(0) = \alpha(\alpha - 1) \dots (\alpha - n + 1).$$

Then (1.8.1) follows from this last expression and (1.8.2) above.

Remark 1.8.1 For any complex number α and non-negative integer n , the binomial number is defined by

$$\binom{\alpha}{n} = \frac{\alpha(\alpha - 1) \dots (\alpha - n + 1)}{n!}.$$

(Notice that this is a degree n polynomial in α). Equation (1.8.1) is then just a generalization of the binomial theorem to any complex power α , since (1.8.1) can be re-written as

$$(1+z)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} z^n.$$

This expansion is an infinite sum, except when α is a positive integer. In this case it is easy to see that $\binom{\alpha}{n}$ is zero when $n > \alpha$, so that the expansion has a finite number of terms and reduces to the regular version of the binomial theorem.

1.9 Problem 08 in section 5.2.

Here we use Taylor series to verify several identities.

- a) $\sin(-z) = \sin(z)$ Substituting $z \rightarrow -z$ into the Taylor series for the sine (see equation (10) in section 5.2 of the book), we obtain:

$$\sin(-z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (-z)^{2n+1} = - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} = -\sin(z).$$

Since $\sin(z)$ is entire, its Taylor series converges everywhere in the complex plane and the equality holds for all z .

- b) $\frac{de^z}{dz} = e^z$ Consider the Taylor expansion for the exponential at the origin

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Differentiating both sides term by term, we get

$$\frac{de^z}{dz} = \sum_{n=1}^{\infty} n \frac{z^{n-1}}{n!} = \sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z.$$

Since e^z is entire, its Taylor series converges everywhere in the complex plane and the equality holds for all z .

- c) $e^{-iz} = \cos(z) - i \sin(z)$ We start with the Taylor expansion for the exponential at the origin

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

and substitute $z \rightarrow -iz$. This yields

$$\begin{aligned} e^{-iz} &= \sum_{n=0}^{\infty} \frac{(-i)^n z^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{(-i)^{2n} z^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(-i)^{2n+1} z^{2n+1}}{(2n+1)!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} - i \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} \\ &= \cos z - i \sin z. \end{aligned}$$

Again, all the series involved converge everywhere, thus the equality holds for all z .

d) $e^{2z} = (e^z)^2$ Using the Taylor expansion for the exponential around the origin, we have

$$e^z e^z = \left(\sum_{i=0}^{\infty} \frac{z^i}{i!} \right) \left(\sum_{j=0}^{\infty} \frac{z^j}{j!} \right) = \sum_{n=0}^{\infty} \left(\sum_{i+j=n} \frac{1}{i!j!} \right) z^n.$$

From the binomial theorem

$$n! \sum_{i+j=n} \frac{1}{i!j!} = \sum_{i+j=n} \frac{n!}{i!j!} = \sum_{i=0}^n \binom{n}{i} = (1+1)^n = 2^n.$$

Thus

$$(e^z)^2 = \sum_{n=0}^{\infty} \frac{2^n}{n!} z^n = e^{2z},$$

which (again) holds for all z .

1.10 Problem 11 in section 5.3.

Consider two power series with a positive radius of convergence (each of them then defines an analytic function in a neighborhood of the origin):

$$a(z) = \sum_0^{\infty} a_k z^k \quad \text{and} \quad b(z) = \sum_0^{\infty} b_k z^k.$$

Let R be the smallest of the radius of convergence; then a and b are both analytic in the open disk $|z| < R$, where the following also applies

$$\frac{d^n a(z)}{dz^n} = \sum_{k=n}^{\infty} \frac{k!}{(n-k)!} a_k z^{k-n} \quad \text{and} \quad \frac{d^n b(z)}{dz^n} = \sum_{k=n}^{\infty} \frac{k!}{(n-k)!} b_k z^{k-n}.$$

Thus

$$a_n = n! \frac{d^n a(0)}{dz^n} \quad \text{and} \quad b_n = n! \frac{d^n b(0)}{dz^n}. \quad (1.10.1)$$

a) If $a(z) = b(z)$ in a neighborhood of the origin, then their derivatives at the origin are all equal.

Therefore (1.10.1) shows that $a_k = b_k$ for all k .

b) If $a(x) = b(x)$ for all x real in an open interval containing origin, then (again) the derivatives of a and b are equal at the origin.² Therefore (1.10.1) shows that $a_k = b_k$ for all k .

²This because to calculate a derivative (once we know it exist) we can take the limit (in the differential quotient defining the derivative) along any line approaching the point where we want the derivative.

1.11 Problem 14 in section 5.3.

Here we consider the initial value problem

$$\frac{d^2 f}{dz^2} + f = 0, \quad \text{where } f(0) = 0 \text{ and } f'(1) = 1. \quad (1.11.1)$$

We use power series to show that, if f is analytic at $z = 0$, then $f(z) = \sin(z)$.

Since f is analytic at the origin, it must have a Maclaurin series representation. We can find this series by computing the derivatives $f^{(n)}(0)$ at the origin, which we do next.

Because f satisfies the differential equation, we have

$$\frac{d^2 f(z)}{dz^2} = -f(z).$$

Taking repeated derivatives of this equation, it is easy to see that we have (can be proved by induction)

$$\frac{d^{2n} f(z)}{dz^{2n}} = (-1)^n f(z) \quad \text{and} \quad \frac{d^{2n+1} f(z)}{dz^{2n}} = (-1)^n \frac{df(z)}{dz},$$

for $n = 0, 1, \dots$. Thus, from the initial conditions at $z = 0$, we have

$$f^{(2n)}(0) = 0 \quad \text{and} \quad f^{(2n+1)}(0) = (-1)^n.$$

Therefore, we can write the solution as

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0) z^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} = \sin z.$$

This finishes the problem.

THE END.