Answers to Problem Set Number 1 for 18.04. MIT (Fall 1999)

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Contents

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1 Problems from the book by Saff and Snider.

1.1 Problem 15 in section 1.1.

The following calculations prove the required results:

$$\begin{cases} i^{4k} & = & (i^4)^k & = & 1^k & = & 1\,, \\ i^{4k+1} & = & (i^{4k})\cdot i & = & 1\cdot i & = & i\,, \\ i^{4k+2} & = & (i^{4k+1})\cdot i & = & i\cdot i & = & -1\,, \\ i^{4k+3} & = & (i^{4k+2})\cdot i & = & (-1)\cdot i & = & -i\,. \end{cases}$$

1.2 Problem 24 in section 1.1.

Write $z = x + iy$, where $x, y \in \mathbf{R}$ and $y > 0$. Then we have $\dfrac{1}{z} = \dfrac{\bar{z}}{z\bar{z}} = \dfrac{(x - iy)}{(x^2 + y^2)}$ and it follows that: $\mathrm{Im}(\dfrac{1}{z}) = \dfrac{-y}{(x^2 + y^2)} < 0$.

1.3 Problem 05 in section 1.2.

Let $z_1 = 1$, $z_2 = -(1/2) + i(\sqrt{3}/2)$ and $z_3 = -(1/2) - i(\sqrt{3}/2)$. Then $|z_1 - z_2| = |(3/2) - i(\sqrt{3}/2)| = \sqrt{3}$, $|z_1 - z_3| = |(3/2) + i(\sqrt{3}/2)| = \sqrt{3}$ and $|z_2 - z_3| = |i\sqrt{3}| = \sqrt{3}$. Since all sides of the triangle have the same length, these points are the vertices of an equilateral triangle.

1.4 Problem 08 in section 1.2.

Let $z = x + iy$ be the cartesian representation of z. Then since $|z - 1| = |(x - 1) + iy| = \sqrt{(x-1)^2 + y^2}$, we have: $|\bar{z} - 1| = |(x - 1) - iy| = \sqrt{(x-1)^2 + y^2} = |z - 1|$.

Graphically, this is just a consequence of the fact that (in the plane) the distance from an arbitrary point A to some point B on a straight line is the same as the distance from $\bar{A}$ to B, where $\bar{A}$ is the reflection of A across the line.

1.5 Problem 02 in section 1.3.

Let $z_j = r_j \exp(i\theta_j)$, for $j = 1, 2, 3$. Then $|z_j| = r_j$ for $j = 1, 2, 3$, so that: $|z_1||z_2||z_3| = r_1 r_2 r_3$. Thus: $|z_1 z_2 z_3| = |r_1 r_2 r_3 \exp(i(\theta_1 + \theta_2 + \theta_3))| = r_1 r_2 r_3 = |z_1||z_2||z_3|$.

1.6 Problem 11 in section 1.3.

Let $z_1 = r_1 \exp(i\theta_1)$ and $z_2 = r_2 \exp(i\theta_2)$, so that $\arg(z_1) = \theta_1$ and $\arg(z_2) = \theta_2$ (modulo integer multiples of 2π, which can always be added to the argument). It follows then that: $\arg(z_1 \bar{z}_2) = \arg(r_1 \exp(i\theta_1) r_2 \exp(-i\theta_2)) = \arg(r_1 r_2 \exp(i(\theta_1 - \theta_2)) = \theta_1 - \theta_2$ (again, modulo multiples of 2π).

1.7 Problem 17 in section 1.3.

By definition, z_1 is parallel to z_2 if and only if $z_1 = \lambda\, z_2$, for some $\lambda \in \mathbf{R}$. Now suppose that $z_1,\ z_2 \neq 0$ and write $z_1 = r_1 \exp(i\theta_1)$ and $z_2 = r_2 \exp(i\theta_2)$. Then $z_1 = \lambda z_2 \iff r_1 = \lambda r_2 \exp(i(\theta_2 - \theta_2))$. Since r_1, r_2 and λ are all real, this last equation can have a solution if and only if $\theta_1 - \theta_2$ is an integer multiple of $2\pi \iff \mathrm{Im}(z_1 \bar{z}_2) = 0$.

1.8 Problem 05 in section 1.4.

We have: $|\exp(x + iy)| = |e^x(\cos y + i \sin y)| = \sqrt{e^{2x}(\cos^2 y + \sin^2 y)} = \sqrt{e^{2x}} = e^x$.

Similarly: $\arg(\exp(x + iy)) = \arg(\exp(x)\exp(iy) = y + 2k\pi$ for $k \in \mathbf{Z}$.

1.9 Problem 13 in section 1.4.

Using the expressions for the sine and the cosine in terms of exponentials, we find:

(a) $\sin^2\theta + \cos^2\theta = (1/4)[(2 - e^{2ix} - e^{-2ix}) + (2 + e^{2ix} + e^{-2ix})] = (1/4)\cdot 4 = 1\,.$

(b)
$$\begin{aligned}
\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 &= (1/4)[(e^{i\theta_1} + e^{-i\theta_1})(e^{i\theta_2} + e^{-i\theta_2}) + (e^{i\theta_1} - e^{-i\theta_1})(e^{i\theta_2} - e^{-i\theta_2})] \\
&= (1/4)[e^{i(\theta_1+\theta_2)} + e^{i(\theta_1-\theta_2)} + e^{-i(\theta_1-\theta_2)} + e^{-i(\theta_1+\theta_2)} \\
&\qquad + e^{i(\theta_1+\theta_2)} - e^{i(\theta_1-\theta_2)} - e^{-i(\theta_1-\theta_2)} + e^{-i(\theta_1+\theta_2)}] \\
&= (1/2)(e^{i(\theta_1+\theta_2)} + e^{-i(\theta_1+\theta_2)}) \\
&= \cos(\theta_1 + \theta_2).
\end{aligned}$$

1.10 Problem 16 in section 1.4.

Since $r > 0$, $\ln r$ is real and we have: $\exp(\ln r + i\theta) = \exp(\ln r)\exp(i\theta) = r\exp(i\theta) = z$.

1.11 Problem 02 in section 1.5.

We proceed by induction. Let $z = re^{i\theta}$. Then, by letting $z_1 = 1$ and $z_2 = z$ in formula (14) of section 1.4 of the book, we find that $z^{-1} = (1/r)e^{-i\theta} = r^{-1}e^{-i\theta}$. Now suppose that $z^{-(n-1)} = r^{-(n-1)}e^{-i(n-1)\theta}$. Then, letting $z_1 = z^{-(n-1)}$ and $z_2 = z$ in the same formula (14), we obtain $z^{-n} = z^{-(n-1)}/z = (r^{-(n-1)}/r)\exp(i(-(n-1)\theta - \theta)) = r^{-n}\exp(-in\theta)$.

1.12 Problem 04 in section 2.1.

(a) Since $|z| = r$, we can write $z = re^{i\theta}$. Then $w(z) = 1/z = (1/r)e^{-i\theta}$, so that $|w| = 1/r$. Clearly the map is a biyection from circle to circle.

(b) Since $\text{Arg}(z) = \theta_0$, we can write $z = re^{i\theta_0}$ (with $0 < r < \infty$). Then $w(z) = 1/z = (1/r)e^{-i\theta_0}$. Thus we have $\text{Arg}(w) = -\theta_0$. Again, it is clear that the map is a biyection from ray to ray.

1.13 Problem 08 in section 2.1.

The answer to this problem can be found in figure 1.13.1 next.

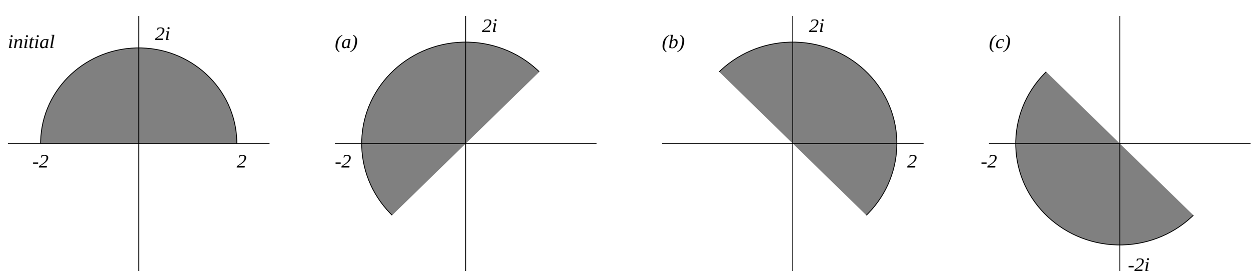

Figure 1.13.1: Images of the semidisk $|z| \leq 2$ and $\text{Im}(z) \geq 0$, by the map $G(z) = e^{i\phi}z$, with: (a) $\phi = \pi/4$, (b) $\phi = -\pi/4$ and (c) $\phi = 3\pi/4$.

1.14 Problem 09 in section 2.1.

The answer to this problem can be found in figure 1.14.1 next.

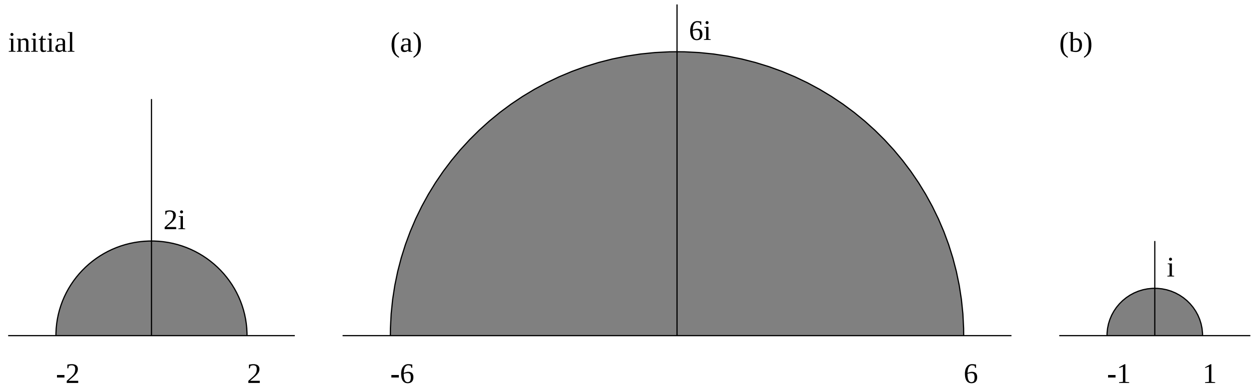

Figure 1.14.1: Images of the semidisk $|z| \leq 2$ and $\text{Im}(z) \geq 0$, by the map $H(z) = \rho z$, with: (a) $\rho = 3$ and (c) $\rho = 1/2$.

2 Other problems.

2.1 Problem 1.1 in 1999.

Statement: Express $\cos(3\theta)$ in terms of $\cos\theta$ using De Moivre's formula.

Solution: Using De Moivre's formula, we find:

$$\begin{aligned}
\cos(3\theta) &= \text{Re}[(\cos\theta + i\sin\theta)^3] \\
&= \text{Re}[\cos^3\theta - 3\cos\theta\sin^2\theta + 3i\sin\theta\cos^2\theta - i\sin^3\theta] \\
&= \cos^3\theta - 3\cos\theta\sin^2\theta\,.
\end{aligned}$$

But $\sin^2\theta = 1 - \cos^2\theta$, thus: $\boxed{\cos(3\theta) = 4\cos^3\theta - 3\cos\theta\,.}$

THE END.