Pset 10 Part I

Problem 1: *Find the critical points of the non-linear autonomous system*

$$x' = 1 - x + y$$
$$y' = y + 2x^2$$

Solution: Critical points occur where $1 - x + y = 0$ and $y + 2x^2 = 0$ Substituting the first equation rewritten as $y = x - 1$ into the second $y + 2x^2 = 0$ we get

$$0 = x - 1 + 2x^2 \Rightarrow x = \frac{1}{2} \text{ or } x = -1.$$

Then $x = \frac{1}{2} \Rightarrow y = -\frac{1}{2}$, and $x = -1 \Rightarrow y = -2$.
Thus, the critical points are $\left(\frac{1}{2}, -\frac{1}{2}\right)$ and $(-1, -2)$.

Problem 2: *Write as equivalent first-order system and find the critical points:*

$$x'' - x' + 1 - x^2 = 0$$

Solution: Let $y = x'$, then $y' = x'' = x' - 1 + x^2$. So the equivalent 2×2 autonomous system is then

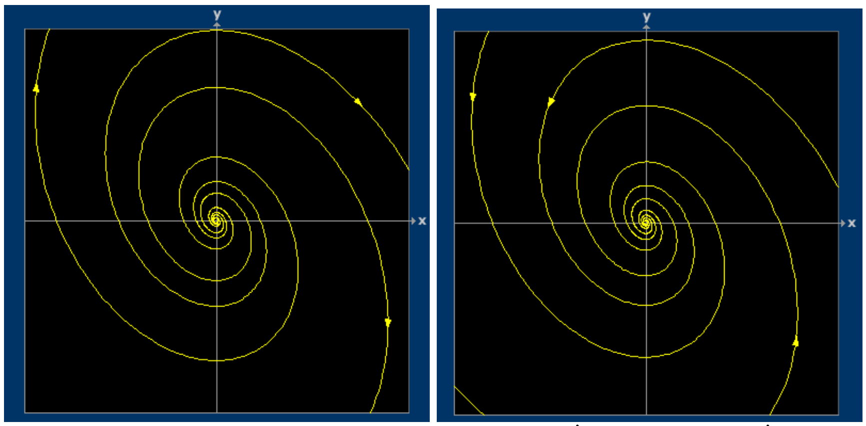
$$x' = y$$
$$y' = y - 1 + x^2$$

Critical points occur when $y = 0$ and $y - 1 + x^2 = 0 \Rightarrow y = 0$ and $x^2 = 1$.
So, the critical points are $(1, 0)$ and $(-1, 0)$.

Problem 3: *In general, what can you say about the relation between the trajectories and the critical points of the system on the left below, and those of the two systems on the right?*

$$\begin{aligned} x' &= f(x,y) \\ y' &= g(x,y) \end{aligned} \qquad a)\ \begin{aligned} x' &= -f(x,y) \\ y' &= -g(x,y) \end{aligned} \qquad b)\ \begin{aligned} x' &= g(x,y) \\ y' &= -f(x,y) \end{aligned}$$

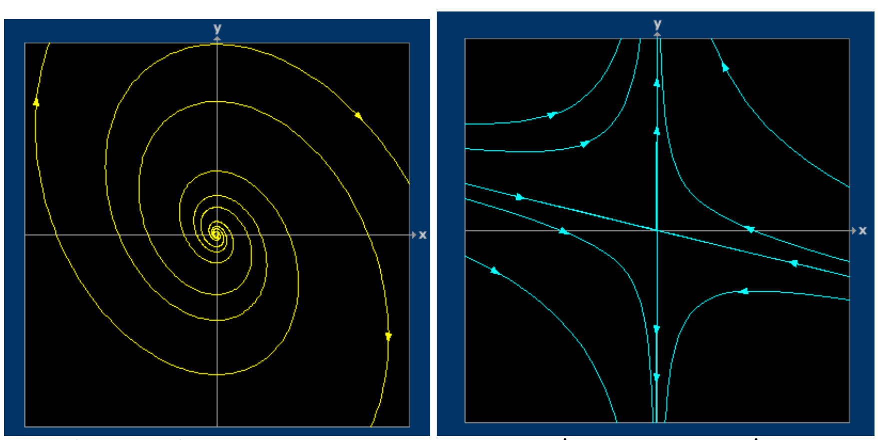
Solution: **(a)** For this system the tangent vector $(-f(x,y), -g(x,y))$ to a trajectory is equal in magnitude but opposite in direction to the tangent vector $(f(x,y), g(x,y))$ to the original system. So the trajectory paths are the same but are traversed in the opposite direction.



The left hand figure is the original system $x' = .6x + y$, $y' = -x$.
The right hand figure is the reversed system $x' = -.6x - y$, $y' = x$.

The critical points are the same for both systems, they occur at $f(x,y) = 0$ and $g(x,y) = 0$.

(b) For this system the tangent vector $(g(x,y), -f(x,y))$ to a trajectory is perpendicular to the tangent vector $(f(x,y), g(x,y))$ to the original system. So the solutions to (b) are the orthogonal trajectories to the original system.



The left hand figure is the original system $x' = .6x + y$, $y' = -1.5x$.
The right hand figure is the orthogonal system $x' = -1.5y$, $y' = .6x + y$.

The critical points occur at $g(x,y) = 0$ and $-f(x,y) = 0$, i.e. they are the same as for the original system.