18.03SC Differential Equations, Fall 2011
Transcript – Lecture 33

Today's lecture is going to be basically devoted to working out a single example of a nonlinear system, but it is a very good example because it illustrates three things which you really have to know about nonlinear systems. I have indicated them by three cryptic words on the board, but you will see at different points in the lecture what they refer to. Each of these represents something you need to know about nonlinear systems to be able to effectively analyze them. In addition, I am not allowed to collect any more work, according to the faculty rules, after today. And, of course, I won't. But, nonetheless, the final will contain material from the reading assignment G7, and I will be touching on all of these today, and 7.4, just a couple of pages of that.

You will see its connection. It discusses the same example we are going to do today. And then I suggest those three exercises to try to solidify what the lecture is about. Instead of introducing the nonlinear system right away that we are going to talk about, I would like to first explain, so that it won't interrupt the presentation later, what I mean by a conversion to a first-order equation.

For that why don't we look at -- Maybe I can do it here. Our system looks like this, dx/dt. For once, I am not writing x' because I want to explicitly indicate what the independent variable is. It is a nonlinear autonomous system, so there is no t on the right-hand side. But this is not a simple linear function, ax + by. It is more complicated, like you had in the predator-prey robin-earthworm problem that you worked on last night. And the other equation will be g of (x, y). This is the system, and explicitly it is nonlinear in general and autonomous, as I have indicated on the right-hand side.

Now, remember that the geometric picture of this was as a velocity field. You made a velocity field out of the vectors whose components were f and g, so that is (f)i + (g)j, but what it looked like was a plane filled up with a lot of vectors pointing different ways according to the velocity at the point was. And side-by-side with that went the solutions. A typical solution was x = x(t), y = y(t) represented as a column vector. And that is a parametric equation. And the geometric picture of that was given by its trajectory.

When you plotted it, it was a trajectory. It had not only the right direction at each point, in other words, but it also had to have the right velocity at each point. In other words, sometimes the point was moving rapidly and sometimes it was moving more slowly along that path. And the way it moved was an important part of the solution. Now, what I plan to do is -- The conversation that I am talking about takes place by eliminating t.

Now, why would one want to do that? t, after all, is an essential part of the solution. It is an essential part of this picture because without t you would not know how long the arrows were supposed to be. You would still know their direction. And, of course, it occurs in here. Now, let's take the first step and eliminate t from the system itself.

As you see, that is very easy to do because the right-hand side has no t in it anyway and the left-hand side has the t only as the denominators of the differential quotients. The obvious way to eliminate t is just to divide one equation by the other. And since usually we think of not dx/dy, but dy/dx, I will vow --

What is dy over dx? Well, if I divide this equation by that equation, dy/dt / (dx/dt), according to the chain rule, or according to commonsense, you cancel the dt's and you get dy/dx. And, on the right-hand side, you get g of (x, y) divided by f of (x, y). But what is that? That is the equation of the first day of the term. By taking that single step, I convert this nonlinear system, which we virtually never find explicit solutions to, into an ordinary first order equation which, in fact, you also don't usually find explicit solutions to, except with our narrow blinders on.

We only get problems at the beginning of 18.03 where there will be an explicit solution to this. I have converted the nonlinear system to a single first order equation to which I can apply the usual first order methods that include hope that I will be able to solve it. Now, what do I lose? Well, what happens to this picture? If you don't have the t in it anymore then you don't have velocity, you don't have arrows with a length to them because the length gives you how fast the point is going. Without t I don't know at any point how fast it is supposed to be going. All I know at any point (x, y) is no longer dx/dt and dy/dt. All I know is dy/dx. In other words, the corresponding picture, if I eliminate t from this picture all that is left is the directions of each of these arrows.

What disappears is their length. And, in fact, since dy/dx is purely a slope, I cannot even tell whether the point is traveling this way or that way. It doesn't make any sense to have the point traveling anymore since there is no time in which it can do its traveling. So that picture gets changed to the direction field. The corresponding picture now, all you do is take each of these arrows, you snip it off to a standard length, being careful to snip off the little pointy end, and it becomes nothing but a line element. And all it has is a slope. What corresponds to the velocity field here is now just our slope field or our direction field.

All we know is the slope at each point. How about the solution? Well, if I have eliminated time, the solution is no longer a pair of parametric equations. It is just an equation involving x and y. I can hope that the solution will be explicit, y = y(x), but you already know from the first days of the term that sometimes it isn't. Let's call that explicit solution. Often you have to settle for it. And sometimes it is not a pain. It is something that is even better. Settle for a solution which looks like this which is y defined implicitly as a function of x. And what does the picture of that look like? Well, the picture of this is now simply an integral curve. It is not a trajectory anymore. It is what we called an integral curve.

I am reminding you of the very first day of the term, or maybe the second day. And all it has at each point the right slope because that is all the field is telling me now. It only has a slope at each point. It doesn't have any magnitude or direction. It has the direction, but it doesn't have direction in the sense of an arrow telling me whether it is going this way or the opposite way. That is the picture with t in it.

These are the corresponding degeneration of that. It is a coarsening, it's a cheapening of it. It is throwing away information. Throw away all the information that had to do with t, and we are back in the first days of the terms with a single first order equations involving only x and y with solutions that involve only x and y with curves that are the graphs of these solutions but again have no t in them. In effect, they are just the paths of the trajectories.

Whereas, the trajectory is the point that shows actually how the point is moving along in time faster or slower at various points. Now, why would one want to lose information? Well, the great gain is that this might be solvable. Whereas, this almost certainly is not. That is a big plus. For example, let's illustrate it on the very simplest possible case. I cannot illustrate it on a nonlinear system very well, or not right now because, in general, nonlinear systems are not solvable. Let's take an easy example. Suppose the system were a linear one, let's say this one that we have talked about before, in fact. That is a simple linear two-by-two system.

This means the derivatives with respect to time. Well, you know that the solutions we talked about in fact, last time, typical solutions that would be something like (x, y) equals a constant times, let's say if $x = \cos(t)$ then y would be the derivative of that, $y = -\sin(t)$. Or, another one would be $c_2 (\sin(t), \cos(t))$.

And those, of course, are circles. They are parametrized circles, so they are circles that go around, in fact, in this direction. And they have a certain velocity to them. Now what do I do if I follow this plan of eliminating t? Well, if I eliminate t directly from the system, what will I get? I will get dy/dx. I divide this equation by that one. -- is equal to -x/y. Oh, well, of course, that is solvable. You were able to solve by 18.01 methods before you ever come into this course. By separation of variables, it is $y\,dy = -x\,dx$, which integrates to be $1/2\, y^2 = -1/2\, x^2 + c$.

And after multiplying through by two and moving things around it becomes $x^2 + y^2$. The nicest way to say it is implicitly, x squared plus y squared equals some positive constant. These are the circles. Now, can I eliminate t? I could also take one of these solutions and eliminate t. If I square this, the best way to do it is not to use our cosines and arcsines and whatnot, which will just get you totally lost.

Square this, square that and add them together. And you conclude that $x^2 + y^2 = 1$. Or, if c_1 is not one it is equal to c_1 squared. You could eliminate t from the solution the way you eliminate t from a pair of parametric equations, and you get it the same way that $x^2 + y^2 = (c_1)^2$, I guess.

In some sense, what we are doing is cycling around to the beginning of the term. In fact, this whole lecture, as you will see, is about cycles of one sort or another. But I keep thinking of that great line of poetry, "in my end is my beginning" or maybe it's "in my beginning is my end". I think both lines occur. But that is what is happening here. This is the beginning of the course and this is the end of the course, and they have this almost trivial relation between them. But notice the totally different methods used for analyzing this, where the t is included, than from analyzing this where there is no t. The goals are different, what you look for are different, everything is different, and yet it is almost the same problem.

I promised you a nonlinear example. I guess it is time to see what that is. It is going to be another predator-prey equation, but one which is, in some ways, simpler than the one I gave you for homework. The predator is going to be x because you are used to that from the robins. I am keeping the same predator-prey variables. And the prey will be y. And now just notice the small difference from what I gave you before. For one thing, I am not giving you specific numbers. I am going to, at the beginning at least, do some of the analysis at the beginning using letters, using parameters. a, b, c, d are always going to be positive constants. I am going to assume that $x' = -ax$.

This represents the predator dying out if there is no prey there, but if there is something to eat that term represents the predator meeting up with the prey and gobbling it up. And b is the coefficient. How about without predictors, the prey will multiply and be fruitful. Unfortunately they get eaten, and so there will be a term that looks like this. Now, there are two basic differences between these simple equations and the slightly more complicated ones you had with the robin-earthworm equations. Namely, in the first place, I am assuming that these guys, let's give them names. I want you to remember which is x and which is y, so we are going to think of these are "sharx". [LAUGHTER]

And these will be food fish, so let's make them "yumfish". The fact that with your robins you had a positive term here. I made this 2x. And now it is minus a times x. What is the difference? The difference is that robins have other things to eat, so even if there are not any worms, a robin will survive. It could eat other insects, grubs, Japanese beetle grubs. I think it can even eat seeds. Anyway, the robins in my garden seem to be pecking at things that don't seem to be insects. And the other is that I am assuming a very naïve growth law. For example, if there are no sharks, how are the food fish fruitful and multiplying?

They multiple exponentially. Now, obviously you cannot have unlimited growth like that. With the worms we added the term minus a constant times y squared to indicate that even worms cannot multiply purely exponentially forever, but ultimately their growth levels off because they cannot find enough organic matter to plow their way through. Those are the two differences. I am not assuming a logistic growth law. This is less sophisticated than the one I gave you for homework. This assumes that sharks have absolutely nothing to eat except these fish, which is not so bad. That's true, more or less.

My plan is now, with this model, let's start the analysis, as you learned to do it. And we are going to run into trouble at various places. And the troubles will then illustrate these three points that I wanted to add to your repertoire of things to do with nonlinear systems when you run into trouble. And it will also increase your understanding of the nonlinear systems. The first thing we have to do is find the critical points. I am going to assume that you are good at this and, therefore, not spend a lot of time detailing the calculations.

I will simply write them on the board. The first equation I am going to write down is $x(-a + by) = 0$. And I assume you know why I am writing that down. And the other equation will be $y(c - dx) = 0$. Those are the simultaneous equations I have to find to find the critical point. The first one is if the product of these is zero, either x is zero or the other factor is. So, from the second equation, if x is zero, y has to be zero also. That is one critical point. Now, if x is not zero then this factor has to be zero which says that $y = a/b$.

And if y is equal to a over b, it is not zero here. Therefore, this factor must be zero which says that $x = c/d$. You see right away that this must be a simpler system because it is only producing two critical points. Whereas, the system that you did for homework had four. Here is one. Well, let's just write them up here. The critical points are zero, zero. That doesn't look terribly interesting, but the other one looks more interesting. It is $(c/d)(a/b)$.

Well, let's take a look first at the zero, zero critical point. The origin, in other words. What does that look like? Well, at the origin, the linearization is extremely easy to do because I simply ignore the quadratic terms, which are the product of two small numbers, where these only have a single small number in it. It is minus ax, cy. In other words, the linearization matrix is (-a, 0; 0, c). Now, I don't think at any point I have ever explicitly told you that I hope you have learned from the homework or maybe your recitation teacher, but for heaven's sake, put this down in your little books, if you have a diagonal matrix, for god's sake, don't calculate its eigenvalues. They are right in front of you.

They are always the diagonal elements. The eigenvalue, you can check this out if you insist on writing the equation, but trust me it is clear. If I, for example, subtract c from the main diagonal, I am going to get a determinant zero because the bottom row will be all zero. The eigenvalues are negative a and c. In other words, they have opposite signs. This is a negative number. That is a positive number --

-- because a, b, c and d are always positive. And, therefore, this is automatically a saddle. You don't have to calculate anything. It is all right in front of you. It must be a saddle and, therefore, unstable because all saddles are. And, in fact, you can even draw the little picture of what the stuff looks like near the origin without even bothering to calculate eigenvectors, although it is extremely easy to do. Just from common sense, these are the sharks and these are the yumfish.

Well, if there are zero yumfish, in other words, if I am on the sharks axis, the axis of sharks, I die out. Well, forget about this side. It's on the positive side. That makes sense. But I die out because the sharks have nothing to eat. Whereas, if I am on the yumfish axis I go this way. I grow because, without any sharks to eat them, the yumfish increase. So it must look like that. And, therefore, the saddle must look like this. The saddle curves hug those and go nearby. Now, the other critical point is the interesting one. And for that the analysis, in order that you don't spend the rest of this period writing a's, b's, c's and d's, I am going to make the simplifying assumption. But it doesn't change qualitatively any of the mathematics. It just makes it a little easier to write everything out. I am going to assume that everything is one.

Well, in fact, even this would be good enough, but let's make everything one. I am going to assume this. And it is only to make the writing simpler. It doesn't really change the mathematics at all. Well, if everything is one, in order to calculate the linearized system, I'd better use the Jacobian. But perhaps it would be better to write out explicitly what the system is. The system now is x' = -x + xy, all the coefficients are one.

And y' = y - xy. What is the Jacobian? Well, the Jacobian is the partial of this with respect to x which is -1 + y, the partial with respect to y which is x, the partial of this with respect to x, which is minus y, and the partial with respect to y, which is 1 - x. But I want to evaluate that at the point (1, 1), which is the critical point. It is the critical point because I have assumed that all these parameters have the value one. And when I do that, evaluating it at (1, 1), I get what matrix? Well, I get (0, 1; -1, 0).

In other words, the linearized system is x' = y and y' = -x. Well, that is just the one we analyzed before in terms of -- It's the one whose solutions are circles. In other words, what we find out is that the linearized system is what geometric type? Saddle? Node? Spiral? None of those. It is a center. The linearized system is a center.

It consists, in other words, of a bunch of curves going round and round next to each other. Concentric circles, in fact. Well, what is wrong with that? Now, we are in deep trouble. We are now in trouble because that is a borderline case. Let me remind you of what the borderline cases are. When we drew that picture, this is from last week's problem set so you have a perfect excuse for forgetting it totally. But I am trying to remind you of it. That is another reason why I am doing this.

Remember that picture I asked you about that appeared on the computer screen? Let's make it a little flatter so that I can have room to write in. This is a certain parabola whose equation you are dying to tell me, but I am not asking. There is the trace, this is the determinant, and the characteristic equation is related to the values of these numbers like plus d equals zero. Then these were spiral synchs. This was the region of spiral sources. That is the abbreviation for sources. These were nodal synchs and these were the nodal sources. And down here were the saddles. And where were the centers? The centers were along this line.

The centers, there weren't many of them, and they were the separation for these two regions. I will put that down. These are the centers. These other borderlines correspond to other things. These are defective eigenvalues, zero eigenvalues and so on. But let's concentrate on just one of them. You will find the others discussed in the notes, GS7. But if you get this idea then the rest is just details. I think it will be perfectly clear.

What is wrong with the center? The answer is that if we are on the center, for example, this system corresponds to the trace being zero and the determinant being plus one. It corresponds to t equals zero and d equals one. This point, in other words. Now, if I wiggle the coefficients of the matrix just a little bit, just change them a little bit, what I am going to do is move off this pink line.

And I might move to here or I might move a little way to there. But, if I do that, I change the geometric type. In other words, being a borderline, a slight change of the parameters can change what it changes to. And, in fact, that is geometrically clear. What does a center look like? A center looks like this, a bunch of curves going around all in the same direction, like concentric circles or maybe ellipses or something like that.

If I deform the picture just a little bit, well, I might change it into something that looks like this where, after they go around they don't quite meet up with where they were to start with. And I am going to get a spiral synch. Or, I might do the deformation by going around once and I'm a little outside of where I was. In which case it's going to be a spiral source. The fact that just changing these curves a little bit can change the picture to this or to that corresponds to the fact that if you are on here with this value of t and d, zero and one, and just change t and d a little bit, you are going to move off into these regions. You might, of course, stay on the pink line. It is not very likely. Where does this leave us?

Well, if the linear system is not stable in the sense that if you change the parameters a little bit it doesn't change the type. This is, after all, only an approximation to the nonlinear system. If in this approximation you cannot really predict the behavior of very well when you change the parameters, then from it you cannot tell what the original system looked like. In other words, the nonlinear system at (1, 1) might be any one of the possibilities, still a center or it might be a spiral synch or it might be a spiral source. Any one of those three is a possibility. It couldn't be a saddle because that is too far away. It couldn't be a node. That is too far away, too. But it could wander into either of these regions and, therefore, the picture you cannot tell which of these three it is just from this type of critical point analysis.

Well, that was Volterra's problem. By the way, the person who introduced these equations and studied them systematically in the way in which we are doing it here was Volterra. And, in fact, he was interested in sharks and food fish, as they were called. What do we do? Well, you have to be smart. What Volterra did was he went to method number one. Let's do it. Volterra said I got my equations $x' = -x + xy$. y prime is equal to, these are the food fish, $y' = y - xy$. Let's eliminate t.

And my problem is, of course, I am trying to determine what type of critical point one, one is. And the method we have used up until now has failed because it gave us a borderline case, which is from that we cannot deduce. He said let's eliminate t. And we get dy/dx equals, I am going to factor, y times one minus x on top. And on the bottom factor x negative one plus y. You could solve that before you came into 18.03, right? This you can separate variables. Let's separate variables. The y's all go on one side, so y goes down here. It is $(y - 1)/y\ dy$ equals --

On the other side the dx gets moved up, one minus x over x dx. Now, of course, you wouldn't dream of using partial fractions on this. It would be illegal because, even though it is a rational function, the quotient of two polynomials, the degree of the top is not smaller than the degree of the bottom. In other words, it is a partial fraction so this degree must be bigger than that one. And it isn't so you would have to first divide. And if you divide by that then there is no point in using partial fractions at all.

Of course, you would have done this by basic instant, I know. What is the solution? It is $y - \ln(y) = \ln(x) - x + c_1$. That is not the final constant I want so I will give it a subscript one to indicate I want more. This is very hard to see what that curve looks like. We can make it look better by exponentiating. If I exponentiate it, going back to high school mathematics, but I know from experience that many of you are not too good at this step, so that is another reason I am doing it, it will be e^y, times, that part is easy because pluses and minuses change into times, times $e^{-\ln(y)}$.

Well, $e^{\ln(y)} = y$. If I put a minus sign in front of that, that sends it into the denominator, so instead of being y it is 1/y. Equals x, $e^{\ln(x)} = x$, e^{-x} is, therefore, times $1 / e^{x}$. And that is times the exponential of c_1, which I will call c_2. And now if I combine them all and put them all on one side it is $x / e^x * y / e^y$ is equal to yet another constant, $1/c_2$. This is my final constant so let's call that c. In other words, the integral curves are the graphs of this equation for different values of the constant c.

Well, of course you've graphed an equation like that all the time. What am I going to do with this stupid thing? Well, I deliberately picked something which involved all your learning up until now. We are now going into 18.02, right? You looked at that in 18.02 you would say this is the contour curve, these are contour curves of the function -- Well, let's write it out the other way. It doesn't make any difference, but you are more likely to have seen the function in this form. x / e^x you won't recognize, but this you will. In fact, we have had it before this term. It is one of the kinds of solutions you can add to second order equations. Times y e^(-y) = c. It is the contour curves of this function.

Let's call that function, let's say, h(x, y). Well, of course you could throw it on Matlab, as you did in 18.02 maybe, and ask Matlab to plot the function. But Volterra didn't have that luxury. He was smart, too. So let's be smart instead. What is the function x? Let's use a neutral variable like u and plot u e^(-u). 18.01. At zero, it has the value zero. When u is small this has approximately the value one and, therefore, it starts out like the function u. As u goes to infinity, you know by L'Hopital's rule that this goes to zero so it ends up like this. Well, what on earth could it possibly do except rise to a maximum? But where is that maximum?

It is easy to see there is a unique maximum just by 18.01. Take the derivative, find out where the maximum point is and you will find, perhaps you have already done it, this is at one. The maximum occurs at one, that is where the derivative is zero and the value there is one times e^(-1). It is 1 / e, in other words. That is what the curve looks like. What do the contour curves of this look like?

Well, if we are looking at the contour curves of the function h, so here is x, here is y, and we want the contour curves of the function h(x, y), what do I know? Where is its maximum? The maximum point of h(x, y) is where? Well, h = x e^(-x) y e^(-y). This has its maximum at one, this has its maximum at one, so where does h(x, y) head? Well, you have two factors. One makes each of them the biggest they can be. The maximum must be exactly at that critical point (1, 1). That is our maximum point.

Let's make it conspicuous. This is the maximum point. It is the point (1, 1). It is the point that makes the function biggest. Now, how about the contour curves? Well, this is the top of the mountain. What is it along the axes? Along the axes, when either x or y is zero, the function has the value zero, so it is biggest there. It has the value zero here. So it is zero value. What are the contour curves? Well, we must have a mountain peak here. This is the unique maximum point. In fact, it is easy to see the contour curves just surround it like that. Now, the reason they cannot be spirals, well, in the first place, you never see contour curves that were spirals. That is a good reason.

It is a mountain. That is the way contour curves of a mountain look. But, all right, that is not a good argument. But notice that along each horizontal line here, I want to know how many times can it intersect the contour curve? That is the same as asking along one of these lines how many times could it intersect? See, this is a graph of the function along this horizontal line. And, therefore, how many times can it intersect one of the contour curves the same number of times that a horizontal line can intersect this curve? Twice. Only once if you are exactly at that height and after that never. But here it can intersect each contour curve only twice which means these things cannot be spirals. Otherwise, they would intersect those horizontal and vertical lines many times instead of just twice.

That is what it looks like. In fact, we can even put in the direction without any effort. We know that the sharks die out without any food fish and the food fish grow. Well, I think this has to be clockwise. Of course, the guys nearby must be going the same way. And, therefore, the guys near them must be going the same way. And it is sort of a domino effect. The direction spreads and there is only one compatible way of making these. They all must be going clockwise. What is happening, actually? At this point there are a lot of sharks but not so many food fish. The sharks eat the few remaining food fish, and now the sharks start to die out because they have nothing to eat. When they are practically gone, the food fish can start growing again and grow and grow.

For a while they grow happily, and then the sharks suddenly start being able to find them again. And the few remaining sharks start eating them, the population of sharks grows, the food fish start dying out again, and the cycle starts all over again. Now, I wanted to talk about the qualitative behavior. This is, in some ways, the most important part of the lecture. I wanted to discuss, just as we did at the beginning of the term, the effect of fishing with nets at a constant rate k.

Just as near the beginning of the term, we talked about fishing a single population. The only difference is, now there are two populations. Both sharks and the food fish. Now, what happens to the equations? Well, the equations start out just as they were. I am now reverting to a and b. You will see why. It's because we need the general coefficients. Plus bxy. But, if we are fishing with nets, then a certain fraction of the sharks in the sea get hauled in by the net.

And say the fishing rate is k. There is a fishing term. We remove minus k times a certain fraction of the sharks in the sea. How about the food fish? Well, we remove them, too. And we don't distinguish between sharks and food fish. This is cy - dxy. But we also remove a certain fraction of them. -ky. What, therefore, is the new system? The new system is therefore x' = -(a + k)x + bxy. And the y' = (c - k)y - dxy.

To solve these, I do not have to go through the analysis. All I have to do is change the numbers, change these coefficients from a --> a + k and c minus k. The old critical point was the point (c/d, a/b). The new critical point with fishing is what? Well, the parameters have been changed by the addition of k. The new critical point is ((c - k)/d, (a + k)/d). What is the effect of fishing? Well, if the old critical point was over here, let's say this is the point c / d and this is the point a / b, that is the old critical point that was over here.

The result of fishing is to lower the value of x and raise the value here. The new critical point is, this gets lowered to (c - k)/d, and this gets raised to (a + k)/b. In other words, the new point is there. The effect of fishing is to -- It does not treat the sharks and the yumfish equally. The effect of fishing lowers the shark population. See, the critical point gives sort of the average shark population. Of course, it cycles around these. But, on the average, this gives how many sharks there are and how many food fish there are.

The new critical point with fishing lowers the shark population and raises the food fish population. That is not intuitive. And, in fact, that was observed experimentally at a slightly different context. And that is why Volterra started working on the problem. I will need three more minutes. The most interesting application of all is not to sharks and food fish. I cannot assume that you are dramatically interested in how many of them there are in the ocean, but you might be more interested in this.

That thing about lowering is called Volterra's principle. Put that in your books. Volterra's principle. Volterra is spelt over there. -- has found more modern applications than sharks. Suppose you consider fish in a pond. You have mosquito larvae which breed in the pond. This happens. And then suddenly there is a plague of mosquitoes and concerned citizens. And this is what happened in the `50s. That was before you were born, but not before I was. What happened was a lot of mosquitoes. Everybody said the mosquitoes breed in the little stagnant ponds so spray DDT on them. DDT them.

Dump it in the ponds. That will kill all the larvae and we won't get bitten anymore. When you put DDT in the pond, as people did not realize at the time because these things were new, of course you kill the mosquitoes, but you also kill the fish because DDT is poisonous to fish. What, in effect, you are doing mathematically is the same as harvesting the fish population with the sharks and the food fish.

The result is, the fish are the predators because they eat the mosquito larvae, big food, there was a certain equilibrium, according to Volterra's principle. With DDT that equilibrium moves to here. In other words, the effect of indiscriminately spraying the pond with DDT is to increase the number of mosquitoes and kill fish. And, in fact, that is exactly what was observed. The same thing was observed with the bird population and insects. Spraying trees for insects to get rid of some pests ends up killing more birds than it does insects and the insects increase. Thanks.