18.03SC Differential Equations, Fall 2011
Transcript – Linear Systems: Matrix Methods

PROFESSOR: Welcome back. So in this session, we're going to use the matrix method to solve this linear system of differential equations. These are x dot equals 6x plus 5y, and y dot equals x plus 2y. So why don't you take a few minutes to write down the system in matrix form and go through the matrix method to solve it. And I'll be right back.

Welcome back. So let's write down this system in matrix form. You would have a vector with entries x and y prime equals a matrix with entries 6, 5, 1, 2 multiplying the column vector xy. So now, we did big part of the work. The matrix method tells us that we need to find the eigenvalues of this matrix to be able to basically diagonalize it and seek eigenvectors so that then we can just read off the solutions and write the solution of the system as a linear combination of the eigenvectors that we found. So let's look for the eigenvalues first.

The eigenvalues would be computed by seeking the determinant of this matrix in this form, 6 minus lambda, 5, 1, 2 minus lambda. We're going to have an equation on lambda, solve for lambda, and the solutions will be our eigenvalues. So the determinant would be 6 minus lambda multiplying 2 minus lambda minus 5, 1 dot 5, equals to 0.

So here, the lambda that lambda gives us a lambda dot squared. We have minus 6 lambda minus 2 lambda, which would be minus 8 lambda. And then, we would have 2 dot 6, which is 12, minus 5, which gives us 7. So quadratic equation in lambda, and you can factorize it and find the solutions, which is lambda 1 equals to 1, lambda 2 equals to 7.

So we're done with the first part. These are our eigenvalues. They're not repeated. They're just completely different and real valued. So now, we're going to look at the eigenvectors associated to each eigenvalue. So first eigenvector would be associated with lambda 1 equals to 1. So we would be solving this system. We would be solving this system with a new matrix, 6 minus 1. I'm going to spell out this one so that 2 minus 1. So this is just our lambda, multiplying an unknown vector with components a1 and a2 equals to zero vector. And basically here, the unknowns are a1 and a2. So this is simply 5, 5, 1, and 1, a1, a2 equals to 0, 0.

So as you saw before, here, basically, we can read off the equation as being 5a1 plus 5a2 equals to 0 and another one which is a1 plus a2 equals to 0. They're the same equations. So really, we just have a1 plus a2 equals to 0. And so our vector V1 could be picked to just have component 1, for example, a1 equals to 1. And its second component would just be minus 1. That would be one pick for our V1. We could normalize this vector if you wanted to. I'm just going to keep it like this for now.

So if we look now for the second negative vector corresponding to the second eigenvalue of 7, I would be looking for the components of these vectors by doing a similar solving for the same thing. And I'm going to spell it out again so that you see where the terms are coming from. It's just 6 minus the value of my lambda, 0, 0. So here, we have 6 minus 7, which is 1, 5. And then, we have 1 and 2 minus 7, which is minus 5. So really, what do you have is an equation minus 1 plus 5a2 in both cases.

So we can pick a value for a1 or a2 and write down a vector V2. And for example, the form of a1 equals to, let's pick a2 equals to 1. And we would have a1 equals to 5, for example. Again, if you wanted an orthonormal basis formed by your V1, V2, you would just normalize these two vectors.

So here, basically, we can then rewrite the solution to the original system as being linear combinations of-- so I'm just going to write it in vector form. The first vector 1, I keep it in V1, V2, that way you see it. And then, I'll go into the component. We'd have V1 exponential of the value of lambda we found that corresponds to V1. So it would be 1 dot t. And then, V2 exponential of the lambda value that corresponds to V2. And then, basically, we just have constants of integration here.

And so the solution to this problem would be linear combination of the vectors by the basis of our eigenvectors and multiplied by the exponentials assigned a value of the eigenvalues that we found when we looked for the eigenvalues of the matrix of the system. So here, just know that like for the 1D problem that we saw before, we're building a solution based on linear combination of lucky guesses that we used.

And in the one equation case, we used a guess of e to lambda dot t in 1D. Here, in this case, we had a guess of a vector in the form of lambda dot t that we use. And then, basically, we just solved for the lambdas, and solved for the V's, and did a linear combination of all the solutions. Like we did before in the 1D case, we solved the lambda. We had different values of lambda. We did a linear combinations of the exponentials.

So that ends this problem. And here, the key is just to go through the method of diagonalizing your matrix. Basically, it's finding the eigenvalues, and then computing the eigenvectors associated with that, and writing your solutions in terms of a linear combination of the solution that you found.