18.03SC Differential Equations, Fall 2011

Transcript – Lecture 26

I just want to remind you of the main facts. The first thing that you have to do is, of course, we are going to have to be doing it several times today. That is the system we are trying to solve. And the first thing you have to do is find a characteristic equation which is general form, although this is not the form you should use for two-by-two, is A - lambda I = 0. And its roots are the eigenvalues.

And then with each eigenvalue you then have to calculate its eigenvector, which you do by solving the system (A - lambda1 I) alpha = 0 because the solution is the eigenvector alpha 1. And then the final solution that you make out of the two of them looks like alpha1 e^(lambda1 t). Of course you do that for each eigenvalue. You get the associated eigenvector.

And then the general solution is made up out of a linear combination of these individual guys with constant coefficients. The lecture today is devoted to the two cases where things do not go as smoothly as they seem to in the homework problems you have been doing up until now. The first one will take probably most of the period. It deals with what happens when an eigenvalue gets repeated. But I think since the situation is a little more complicated than it is where the case of a characteristic root gets repeated in the case of a second-order equation as we saw it, you know what to do in that case, here there are different possibilities. And I thought the best thing to do would be to illustrate them on an example. So here is a problem.

It came out of a mild nightmare, but I won't bore you with the details. Anyway, we have this circular fish tank. It is a very modern fish tank. It is divided into three compartments because one holds Siamese fighting fish and one goldfish, and one-- They should not eat each other. And it is going to be a simple temperature problem. The three actual compartments have to be kept at different temperatures because one is for tropical fish and one is for arctic fish and one is for everyday garden variety fish. But the guy forgets to turn on the heater so the temperatures start out what they are supposed to be, tropical, icy, and normal. But as the day wears on, of course, the three compartments trade their heat and sort of tend to all end up at the same temperature.

So we are going to let (x)i equal the temperature in tank i. Now, these are separated from each other by glass things. Everything is identical, each has the same volume, and the same glass partition separates them out and no heat can escape. This is very well-insulated with very double-thick Thermopane glass or something like that. You can see in, but heat cannot get out very well. Heat essentially is conducted from one of these cells to the other. And let's assume that the water in each tank is kept stirred up because the fish are swimming around in it. That should be a pretty decent way of stirring a fish tank. The question is how do each of these, as a function of time, and I want to know how they behave over time, so find these functions.

Well, we are going to find them in solutions to differential equations. And the differential equations are not hard to set up. They are very much like the diffusion equation you had for homework or the equations we studied in the beginning of the term. Let's do one carefully because the others go exactly the same way. What determines the flow, the change in temperature? Well, it is the conductivity across the barriers. But there are two barriers because heat can flow into this first cell, both from this guy and it can flow across this glass pane from the other cell. We have to take account of both of those possibilities. It is like in your homework. The little diffusion cell that was in the middle could get contributions from both sides, whereas, the two guys on the end could only get contribution from one.

But here, nobody is on the end. It is circular table. Everyone is dying equally. Everybody can get input from the other two cells. x1' is some constant of conductivity times the temperature difference between tank three and tank one. And then there is another term which comes from tank two. So a times tank two minus the temperature difference, tank two minus tank one. Let's write this out. Remember there will be other equations, too. But instead of doing this, let's do a more careful job with this first equation. When I write it out, remember, the important thing is you are going to have x1, x2, x3 down the left, so they have to occur in the same order on the right in order to use these standard eigenvalue techniques.

The coefficient of x1 is going to be minus a x1 and then another minus a x1. In other words, it is going to be minus 2 ax1. And then the x2 term will be plus a x2. And the x3 term will be plus a x3. Well, you can see now that is the equation for x1 prime in terms of the other variables. But there is symmetry. There is no difference between this tank, that tank, and that tank as far as the differential equations are concerned. And, therefore, I can get the equations for the other two tanks by just changing 1 to 2, just switching the subscripts. When I finally do it all, the equations are going to be, I will write them first out as a system.

Let's take a = 1 because I am going to want to solve them numerically, and I want you to be able to concentrate on what is important, what is new now and not fuss because I don't want to have an extra a floating around everywhere just contributing nothing but a mild confusion to the proceedings. So x1 prime, I am going to take a equal 1 and simply write it -2 x1 + x2 + x3. And so now what would the equation for x2' be?

Well, here x2 plays the role that x1 played before. And the only way to tell that x1 was the main guy here was it occurred with a coefficient negative 2, whereas, the other guys occurred with coefficient 1. That must be what happens here, too. Since x2' is our main man, this is minus x2 and this must be x1 here plus x3. And finally the last one is no different, x3 prime is x1 plus x2.

And now it is the x3 that should get negative 2 for the coefficient. There is a perfectly reasonable-looking set of equations. Just how reasonable they are depends upon what their characteristic polynomial turns out to be. And all the work in these problems is trying to find nice models where you won't have to use Matlab to calculate the roots, the eigenvalues, the roots of the characteristic polynomial.

So we have to now find the characteristic polynomial. The matrix that we are talking about is the matrix, well, let's right away write|A - lambda I|. I cannot use the trace and determinant form for this equation because it is not a two-by-two matrix. It is a three-by-three matrix. I have to use the original form for the characteristic equation. But what is this going to be? Well, what is A? A is minus 2. I am going to leave a little space here. 1, 1, 1 minus 2, 1. And finally 1, 1, negative 2. I subtract lambda from the main diagonal, minus -2 - lambda, minus 2 minus lambda, minus 2 minus lambda. And now that equals zero is the characteristic equation.

The term with the most lambdas in it is the main diagonal. That is always true, notice. Now, each of these I would be happier writing lambda plus 2, so there would be a negative sign, negative sign, negative sign. The product of three negative signs is still a negative sign because three is an odd number. So it is minus the principle term. The product of these three is -(lambda + 2)^3. Now, the rest of the terms are going to be easy. There is another term 1 times 1 times 1, another term 1 times 1 times 1. So to that I add 2, 1 and 1 for those two other terms. And now I have the three going in this direction, but each one of them has to be prefaced with a minus sign. What does each one of them come to?

Well, this is minus 2 minus lambda when I multiply those three numbers together. And so are the other guys. This is 1 times 1 times minus 2 minus lambda, the same thing. There are three of them. Minus because they are going this way, minus 3 because there are three of them, and what each one of them is is negative 2 negative lambda. That is equal to zero, and that is the characteristic equation.

Now, it doesn't look very promising. On the other hand, I have selected it for the lecture. Simple psychology should tell you that it is going to come out okay. What I am going to do is expand this. First imagine changing the sign. I hate to have a minus sign in front of a lambda^3, so let's make this plus and we will make this minus and we will make this plus. I will just change all the signs, which is okay since it is an equation equals zero. That doesn't change its roots any.

And now we are going to expand it out. What is this? (lambda + 2)^3. Lambda cubed plus, and don't get confused because it is this 2 that will kill you when you use the binomial theorem. If there is 1 here everybody knows what to do. If there is an A there everybody knows what to do. It is when that is a number not 1 that everybody makes mistakes, including me. The binomial coefficients are 1, 3, 3, 1 because it is a cubed, I am expanding.

So it is lambda^3 + 3 lambda^2 * 2 + 3 * lambda * 2^2 + 2^3. And now we have the other term. All that is plus because I changed its sign. The next thing is negative 2.

And then the last thing is +3*(-2 - lambda). Let's keep it. So what is the actual characteristic equation? Maybe I can finish it. I should stay over here instead of recopying all of it. Well, there is a lot more work to do. Let's see if we can at least write down the equation. What is it? It is lambda^3. What is the lambda^2 term? It is six and that is all there is. How about the lambda term? Well, we have 12 lambda minus 3 lambda which makes plus 9 lambda. That looks good but constant terms have a way of screwing everything up. What is the constant term? It is A minus 2 minus 6.

Zero. The constant term is zero. That converts this from a hard problem to an easy problem. Now it is a cinch to calculate the stuff. Let's go to this board and continue the work over here. The equation is lambda^3 + 6 lambda^2 + 9 lambda = 0. It is very easy to calculate the roots of that. You factor it. Lambda is a common factor. And what is left? lambda^2 + 6 lambda + 9. That is the sort of thing you got all the time when you were studying critical damping. It is the (lambda + 3)^2. Lambda squared plus 6 lambda plus 9 equals zero.

So the eigenvalues, the roots are what? Well, they are lambda equals zero from this factor and then lambda = -3. But what is new is that the minus 3 is a double root. That is a double root. Now, that, of course, is what is going to cause the trouble. Because, for each one of these, I am supposed to calculate the eigenvector and make up the solution. But that assumed that I had three things to get three different solutions. Here I have only got two things. It is the same trouble we ran into when there was a repeated root. We were studying second or third order differential equations and the characteristic equation had a repeated root. And I had to go into a song and dance and stand on my head and multiply things by t and so on. And then talk very hard arguing why that was a good thing to do to get the answer. Now, I am not going to do the same thing here.

Instead, I am going to try to solve the problem instead. Let's get two points by at least doing the easy part of it. lambda = 0. What am I supposed to do with lambda equals zero? I am looking for the alpha that goes with that. And I find that eigenvector by solving this system of equations. Let's write out what that system of equations is. Well, if lambda is zero, this isn't there. It is just the matrix A alpha = 0. And the matrix A is, I never even wrote it anywhere. I never wrote A. I thought I would get away without having to do it, but you never get away with anything. It's the principle of life. That is A.

If I subtract zero from the main diagonal, that doesn't do a great deal to A. And the resulting system of equations is those same things, except you have the a1's there, too. There is one. a1 - 2a2 + a3 = 0. I am just subtracting zero from the main diagonal so there is nothing to do. a2 minus 2 a3 equals zero. Now I am supposed to solve those. Of course we could do it. Well, how do you know how to solve a system of three linear equations? Well, elimination. You can always solve by elimination. Now we are much more sophisticated than that. You all have pocket calculators so you could use the inverse matrix, right? No. You cannot use the inverse matrix.

What will happen if you punch in those coefficients and then punch in A inverse. What answer will it give you? 0, 0, 0. No, I am sorry. It won't give you any answer. What will it say? It will say I cannot calculate the inverse to that matrix because the whole purpose of this exercise was to find a value of lambda such that this system of equations is dependent. The coefficient determinant is zero and, therefore, the coefficient matrix does not have an inverse matrix. You cannot use that method. In other words, the inverse matrix will never work in these problems because the system of equations you will be trying to solve is always a non-independent system.

And, therefore, its determinant is always zero. And, therefore, there is no inverse matrix because the determinant of the coefficient is zero. All you can do is use elimination or physical insight and common sense. Now, because I teach differential equations everybody assumes, mistakenly, as I think, that I really know something about them. I get now and then graduate students, not in mathematics, but some obscure field of engineering or whatever drift into my office and say I see you teach differential equations. Do you have a minute here? And before I can say no they write their differential equation on the board. And almost invariably it is nothing I have ever seen before. And they so look at me hopefully and expectantly.

So what do I ask them? I don't ask them what they have tried. What I ask them is where did this come from? What field did it come from? Because each field has its own little tricks. It gets the same differential equations all the time and has its own little tricks for solving them. You should do the same thing here. Well, of course we can solve this. And by now most of you have solved it just by inspection, just by sort of psyching out the answer. But a better way is to say look, suppose we had the solution, what would the solution look like? Well, it would look like (a_1, a_2, a_3), whatever the values of those variables were which gave me the solution to the equation, times e^{0t}. But what is this?

$e^{0t} = 1$ for all time. And, therefore, this is a constant solution. What I am asking is to find a constant solution. Now, can I, by inspection, find a constant solution to this? If so it must be the one. Well, there is an obvious constant solution. All the cells have the same temperature. If that is true then there is no reason why it should ever change as time goes on.

The physical problem itself suggests what the answer must be. You don't have to solve equations. In other words, any constant like (1, 1, 1). Well, could it be (20, 20, 20)? Yeah, that is a constant multiple of (1, 1, 1). That is included. My basic constant solution, therefore, is simply (1, 1, 1) e^{0t}. You don't have to include e to the 0t because it is one. Now, just to check, is (1, 1, 1) a solution to these equations?

It certainly is. 1 plus 1 minus 2 is zero in every case. The equations are essentially the same, except they use different variables. By inspection or, if you like, by elimination, but not by finding the inverse matrix you solve those equations. And we have our first solution. Now let's go onto the second one. For the second one, we are going to have to use the eigenvalue lambda equals negative 3.

And now what is the system of equations? Well, now I have to take this and I have to subtract negative 3 from the diagonal elements. $-2 - (-3) = +1$, right? Got that? Each of the diagonal elements, after I subtract minus 3 turns into plus 1. And, therefore, the system becomes, the system I have to solve is $a_1 + a_2 + a_3 = 0$. And what is the second equation? Symmetry is preserved. All the equations are essentially the same, except for the names of the variables so they all must give you the same thing after I subtract minus 3 from the main diagonal. Well, that is what we call a dependent system of equations.

All I have is the same equation repeated twice, but I still have to solve it. Now, what you see is that there are lots of solutions to this. Let me write down one of them. For example, suppose I made $a_1 = 1$ and I made a_2 a 0, then $a_3 = -1$. So here is a solution. That is the eigenvector. And with it, I can make the solution by multiplying by e^{-3t}. There is a solution. But that is not the only alpha I could have chosen. Suppose I chose this one instead. Suppose I kept this 1, but this time made a_3 zero. Well, in that case, there would be $a_2 = -1$.

Now, is this essentially different from that one? It would still be multiplied by e^{-3t}, but don't be fooled by the e to the minus 3t. That is our scalar. That is not what is essential. What is essential is the content of these two vectors. Is either one a multiple of the other? The answer is no. Therefore, they are independent. They are pointing in two different directions in three space, these two vectors.

And, therefore, I have two independent solutions just by picking two different vectors that solve those three equations. This is also a solution. If I call this the eigenvector alpha 1, then I ought to call this one the alpha 2. Hey, we can keep on going through this. Why not make the first one zero? Well, what would happen if I made the first one 0, and then 1, and minus 1? The answer is this one is no longer independent of those two.

I can get it by taking a combination of those two. Do you see what combination I should take? This one minus that one. This guy minus that guy gives me that guy, isn't that right? 1 minus 1, 0 minus minus 1, minus 1 minus 0. This is not a new one. It looks new, but it is not. I can get it by taking a linear combination of these two. It is not independent delta. And that would be true for any other possible solution you could get for these equations. Once you found two solutions, all the others will be linear combinations of them. Well, I cannot use that one. It is not new. And the general solution, therefore, will be a combination, c1 times that one plus a constant times this one.

Plus the first one that I found c3 times (1, 1, 1) e^(0t), which I don't have to write in. That is the general solution to the system, (x1, x2, x3). What happens as time goes to infinity? Regardless of what the values of these two C's this term goes to zero, that term goes to zero and what I am left with is a constant solution. So all of these solutions tend to be the solution where all the cells are at the same temperature. Well, of course there must be some vocabulary word in this. There is. There are two vocabulary words. This is a good eigenvalue. There are also bad eigenvalues.

This is a good repeated eigenvalue, but good is not the official word. An eigenvalue like this, which is repeated but where you can find enough eigenvectors, if lambda is a repeated eigenvalue, it occurs multiply in the characteristic polynomial as a root. But you can find enough independent eigenvectors -- Forget the "but." -- to make up the needed number of independent solutions. For example, if it is repeated once, that is it occurs doubly then somehow I have got to get two solutions out of that as I was able to here. If it occurred triply, I have got to get three solutions out of it.

I would look for three independent eigenvectors and hope I could find them. That is the good case because it tells you how to make up as many solutions as you need. And this kind of eigenvalue is called in the literature the complete eigenvalue. Now, how about the kind in which you cannot? Well, unfortunately, all my life I have called it incomplete, which seems to be a perfectly reasonable thing to call it. However, terminology changes slowly over time. The notes, because I wrote them, call it an incomplete eigenvalue. But the accepted term nowadays is defective. I don't like that.

It violates the "eigenvalues with disabilities act" or something. But I have to give it to you because that is the word I am going to try to use from now on, at least if I remember to use it. It would be the word, for example, used in the linear algebra course 18.06 "plug, plug," defective otherwise. A defective eigenvalue is one where you can get one eigenvector. If it is double, for example, if it a double eigenvalue. It is defective if you can get one eigenvector that goes with it, but you cannot find an independent one. The only other ones you can find are multiples of the first one. Then you are really in trouble because you just don't have enough solutions that you are supposed to get out of that, and you have to do something.

What you do is turn to problem two on your problem set and solve it because that tells you what to do. And I even give you an example to work. Problem two, that little matrix has a defective eigenvalue. It doesn't look defective, but you cannot tell. It is defective. But you, nonetheless, will be able to find two solutions because you will be following instructions. Now, the only other thing I should tell you is one of the most important theorems in linear algebra, which is totally beyond the scope of this course and is beyond the scope of most elementary linear algebra courses as I have taught around the country but, of course, not at MIT. But, nonetheless, it is the last theorem in the course. That means it is liable to use stuff. The theorem goes by different names. Sometimes it is called the principle axis theorem.

Sometimes it is called the spectral theorem. But, anyway, what it says is, if A is a real end-by-end matrix which is symmetric, you know what a symmetric matrix is? The formal definition is it is equal to its transpose. What that means is if you flip it around the main diagonal it looks just the same as before. Somewhere on this board, right there, in fact, is a symmetric matrix. What happened to it? Right here was the symmetric matrix. I erased the one thing which I had to have. (-2, 1, 1); (1, -2, 1); (1, 1, -2) That was our matrix A. The matrix is symmetric because if I flip it around the diagonal it looks the same as it did before.

Well, not exactly. The ones are sort of lying on their side, but you have to take account of that. Is that right? The twos are backward. Well, you know what I mean. Put that element there, this one here, that one there. Exchange these two. Notice the diagonal elements don't all have to be minus 2 for that. No matter what they were, they are the guys that aren't moved when you do the flipping.

Therefore, there is no condition on them. It is these other guys. Each guy here has to use the same guy there. This one has to be the same as that one, and so on. Then it will be real and symmetric. If you have a matrix that is real and symmetric, like the one we have been working with, the theorem is that all its eigenvalues are complete. That is a very unobvious theorem. All its eigenvalues are automatically complete. And it is a remarkable fact that you can prove that purely generally with a certain amount of pure reasoning no calculation at all. But it has to be true, and it is true. You will find there are whole branches of applied differential equations.

You know, equilibrium theory, all the matrices that you deal with are always symmetric. And, therefore, this repeated eigenvalues is not something you have to worry about, finding extra solutions. Well, I guess that is the end of the first part of the lecture. I have a third of it left. Let's talk fast. I would like to, with the remaining time, explain to you what to do if you were to get complex eigenvalues.

Now, actually, the answer is follow the same program. In other words, if you solve the characteristic equation and you get a complex root, follow the program, calculate the corresponding complex eigenvectors. In other words, solve the equations. Everything will be the same except that the eigenvectors will turn out to be complex, that is will have complex entries. Don't worry about it. Then form the solutions. The solutions are now going to look once again like alpha e^((a + bi)t). This will be complex and that will be complex, too. This will have complex entries. And then, finally, take the real and imaginary parts.

Those will be real and they will give real and two solutions. In other words, the program is exactly like what we did for second-order differential equations. We used the complex numbers, got complex solutions. And then, at the very last step, we took the real and imaginary parts to get two real solutions out of each complex number. I would like to give you a simple example of working that out. And it is the system x' = x + 2y.

And $y' = -x - y$. Because it is springtime, it doesn't feel like spring but it will this weekend as it is getting warmer. And since, when I am too tired to make up problem sets for you late at night, I watch reruns of Seinfeld. I am from New York. It is just in my bloodstream. Of course, the most interesting character on Seinfeld is George. We are going to consider Susan who is the girlfriend who got killed by licking poison envelopes. And George carried on their love affair until Susan was disposed of by the writers by this strange death.

And we are going to consider x is modeling Susan's love for George. That is x. And George's love for Susan will be y. Now, I don't mean the absolute love. If x and y are zero, I don't mean that they don't love each other. I just mean that that is the equilibrium value of the love. Everything else is measured as departures from that. So (0, 0) represents the normal amount of love, if love is measured. I don't know what love units are. Hearts, I guess. Six hearts, let's say. Now, in what sense does this model it?

This is a normal equation and this is a neurotic equation. That is why this is George and this is Susan who seemed very normal to me. Susan is a normal person. When y is positive that means that George seems to be loving her more today than yesterday, and her natural response is to be more in love with him. That is what most people are. If y is negative, hey, what's the matter with George? He doesn't feel so good.

Maybe there is something wrong with him. She gets a little mad at him and this goes down. $x' < 0$. And the same way why is this positive? Well, again, it is a psychological thing, but all the world loves a lover. When Susan is in love, as she feels x is high, that makes her feel good. And she loves everything, in fact. Not just George. It is one of those things. You all know what I am talking about. Now, George, of course, is what makes the writers happy. George is neurotic and, therefore, is exactly the opposite. He sees one day that he feels more in love with Susan than he was yesterday.

Does this make him happy? Not at all. Not at all. It makes y prime more negative. Why? Because all he can think of is, my God, suppose I am really in love with this girl? Suppose I marry her. Oh, my God, 40 years of seeing the same person at breakfast all the time. I must be crazy. And so it goes down. Here is our neurotic model. The question for differential equations is, what do the solutions to that look like?

In other words, how does, in fact, their love affair go? Now, there is a reason why the writers picked that model, as you will see. It means they were able to get a year's worth of episodes out of it. And why is that so? Well, let's solve it. The characteristic equation is lambda squared. The matrix that governs this system is A equals (1, 2; -1, -1). The trace of that matrix, the sum of the diagonal elements is zero. There is the zero lambda here. The determinant, which is the constant term, is negative 1, minus negative 2, which is plus 1. So the characteristic equation, by calculating the trace and determinant is $\lambda^2 + 1 = 0$.

The eigenvalues are plus and minus i. Now, you don't have to pick both of them because the negative one lead to essentially the same solutions but with negative signs. Either one will do just as we solved second order equations. The system for finding the eigenvectors, well, we are going to have to accept the complex eigenvector. What is the system going to be? Well, I take the matrix and I subtract i.

We will use i. Subtract i from the main diagonal. So the system is $(1 - i)a1 + 2a2 = 0$. And let's, for good measure, write the other one down, too. It is $-a1 - (1 - i) a2$.

Then what is the solution? Well, you get the solution the usual way. Let's take a1 = 1. Then what is a2? $a2 = (1 - i)/2$ from the first equation. So the complex solution is $(1 - i)/2 e^{(it)}$. Now you have to take the real and imaginary parts of that. This is the only part which technically I would not trust you to do without having someone show you how to do it. What do you do? Well, of course, you know how to separate the real and imaginary parts of that. It is the first thing is to separate the vectors.

I don't know how to explain this. Just watch. The real part of it is 1, one-half. It should be negative 1, so minus this plus that because I didn't put that on the right side. It is $(1, -1/2) + i(0, 1/2)$. Anybody want to fight? 1 plus i times 0 minus one-half plus one-half times i. You saw how I did that? Okay. When you do these problem you do it the same way, but don't ask me to explain what I just did. Here it is $\cos(t) = i(\sin(t))$.

And so the real part will give me one solution. The imaginary part will give me another. Since I have a limited amount of time, let's just calculate the real part. What is it? Well, it is $(1, -1/2)\cos(t) - (0, 1/2)\sin(t)$ Now, what solution is that? This is (x, y). Take the final step. It doesn't have to look like that. $x = \cos(t)$. Do you see that? $x = \cos(t) + 0*\sin(t)$. What is y?

$y = -1/2 \cos(t) - 1/2 \sin(t)$. Now, you may have the pleasure of showing eliminating t. You get a quadratic polynomial in x and y equals zero. This is an ellipse. As t varies, you can see this repeats its values at intervals of 2pi, this gives an ellipse. And if you want to use a little computer program, linear phase, this is not in the assignment, but the ellipses look like this and go around that way. And that is the model of George and Susan's love. x, Susan. y, George. They go round and round in this little love circle, and it stretches on for 26 episodes.