

18.03SC Differential Equations, Fall 2011

Transcript Matrix Exponentials

PROFESSOR: Welcome to this recitation on matrix exponential. So here, we're given matrix A with entries $\begin{pmatrix} 6 & 5 \\ 1 & 2 \end{pmatrix}$. And we're asked to compute the matrix exponential, e^{At} , and to use it to solve the initial value problem $u'(t) = Au(t)$, where here, u are basically vectors with initial condition, $u(0) = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$. So why don't you pause the video, work through the problem? And I'll be right back.

Welcome back. So first, to go ahead and compute the matrix exponential, we need to identify the eigenvalues of matrix A and its eigenvectors. So this is a matrix-- I'll just rewrite here-- that we saw before. And its eigenvalues are again, solution of $6 - \lambda$, $5 - \lambda$, $1 - \lambda$, $2 - \lambda$ equals to 0, which gives us $6 - \lambda$, $2 - \lambda$ minus 5 equals to 0 , $\lambda^2 - 8\lambda + 12$ equals to 0. So you can verify that the eigenvalues would be 1 and 7. $\lambda = 1$ equals to 1. And $\lambda = 2$ equals to 7.

So now, we need to seek the eigenvectors associated to each one of the eigenvalues. So the idea here is to basically move toward a diagonalization of the matrix A . So let's seek the eigenvectors. And here, I'm just going to give them to you, and you can verify the calculation. And this calculation was performed in a previous recitation.

So the eigenvectors, v_1 associated to the eigenvalue $\lambda = 1$ was, for example, $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. And the other one that we found, again, this is one form of the eigenvector, was $\begin{pmatrix} 5 \\ 1 \end{pmatrix}$. So these are from the notes of a previous recitation. So you can verify that these are the two eigenvectors.

And from this point, then we can rewrite this solution, if you recall. I'm just going to go through the steps toward getting to the definition of the exponential matrix. So here, if we didn't know anything about the exponential matrix, we would be able to write the solution as $C_1 e^{t} v_1 + C_2 e^{7t} v_2$, which basically gives us here, if I write it in this form, for example, an exponential t minus exponential $7t$ and an exponential $5t$ multiplied by the entry of this vector, an exponential $7t$ here multiplying C_1, C_2 .

So this is where the idea of the matrix exponential comes from. We're basically introducing the matrix $\phi(t)$ for which we can write $u(t) = \phi(t) u(0)$. So $\phi(t)$ would then be equal to this matrix. But what we want is to be able to solve an initial value problem for which e^{At} applied to our initial conditions would give us back our initial condition. So we're seeking for a form for this exponential matrix that would allow us to do this.

So the way that we define the matrix exponential give us e^{At} . Now, I'm going to the proof, but we're just going to check it together, multiplied by $\phi(0) = I$. So let's check that if we use this form of the matrix exponential, we would have $e^{A \cdot 0} = I$. We will have that at 0 applied to $u(0)$. We have $\phi(0) = I$ applied to $u(0)$. This is a matrix with its inverse, which gives us the identity. And so basically, this gives us back $u(0)$.

I mean you don't need to do that when you're asked to find the matrix exponential. But just to remember where it's coming from, you write down your system in matrix form. You identify the matrix ϕ of t . And then you recall why you want the matrix exponential to have this form, basically to be able to solve initial value problems for which the value u of t is projected to u of 0 when we take t equals 0 for the matrix exponential.

So now let's go back to our problem. So let's compute this matrix exponential. We have ϕ of t . So now from this formula, we know that we need ϕ of 0 . So that gives us, basically, exponential of 0 , 5 , minus 1 , and 1 . We need to find its inverse. So recall that the inverse of a two-by-two matrix is basically just the determinant, minus B minus C and reversing the diagonal entries. So we can just apply this to get our ϕ of 0 minus 1 . So here, our determinant is basically 1 plus 5 , which is 1 over 6 . And then the entries are simply 1 , 1 , minus 5 , and 1 .

So now, we're just left with the multiplication of two matrices to get our matrix exponential. So our matrix exponential would give us this one sixth. And we now have to multiply the entries. So I'm not going to rewrite everything. I'm just going to use this space here. So we have exponential t multiplying 1 plus 5 exponential $7t$. Then, we have exponential t dot minus 5 for this entry. 5 exponential t multiplying our $7t$. Then for the second entry, we basically have minus exponential t exponential $7t$ minus exponential t minus 5 and exponential $7t$.

So we're done with the matrix exponential. So now we were asked to solve for the initial value problem with initial condition 4 and 1 . So how do we go about that? Well, recall that I just reminded you what did we want to use this matrix exponential for. And what we wanted it for is to be able to basically project an initial condition into a solution u of t , t times later. And we constructed this matrix to be able to basically give us this solution by just multiplying the matrix by the initial value vector.

So basically, to find the solution of this initial value problem, we simply need to multiply this matrix by the initial vector that we were given. And I'm just going to write it here to not have to rewrite everything. And it was 4 and 1 . And this is u of 0 . So let me just do a dash here just so that we can do the computation. And we would end up with a solution-- I'm going to keep it in matrix form for now.

So we end up with 4 exponential t minus 5 exponential t , so minus 1 exponential t . And we have a one sixth. Here, 5 exponential $7t$, so we have 20 , plus 5 , so 25 , exponential $7t$. Then for the second entry of the vector solution, we have minus exponential here minus 4 that we add to a 5 , and here, a 7 multiplied by 4 that we add to a 1 . So we have basically plus 5 exponential $7t$. And that basically gives us one way of writing this solution.

And we can split this down, if we will, into two vectors, plus t , minus 1 , 1 exponential $7t$, 25 , 5 . And this form is as valid. Yes, thank you. So that ends the laborious calculations. But basically, the key point here was just to remember where is the matrix exponential coming from, basically, from the eigenvalues and eigenvectors of the original matrix present in the system, and where is the definition coming from, why do we define it as ϕ of t ϕ minus 1 of 0 , and how to use it then to give the solution to an initial value problem. So that ends this recitation.

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