Inhomogeneous Case: Variation of Parameters Formula

The fundamental matrix $\Phi(t)$ also provides a very compact and efficient integral formula for a particular solution to the inhomogeneous equation $\mathbf{x}' = A(t)\mathbf{x} + \mathbf{F}(t)$. (presupposing of course that one can solve the homogeneous equation $\mathbf{x}' = A(t)\mathbf{x}$ first to get Φ.) In this short note we give the formula (with proof!) and one example.

Variation of parameters: (solving inhomegeneous systems)
(H) $\mathbf{x}' = A(t)\mathbf{x} \rightsquigarrow \Phi(t)$ = fundamental matrix
(I) $\mathbf{x}' = A(t)\mathbf{x} + \mathbf{F}(t)$
Variation of parameters formula for solution to (I) (just like order 1 DE's):

$$\mathbf{x} = \Phi \cdot \left(\int \Phi^{-1} \cdot \mathbf{F}\, dt + \mathbf{C} \right).$$

proof (remember this)
General homogeneous solution: $\mathbf{x} = \Phi \cdot \mathbf{c}$ for a constant vector $\mathbf{c}$.
Make c variable $\rightsquigarrow$ trial solution $\mathbf{x} = \Phi \cdot \mathbf{v}(t)$.
Plug this into (I): $\mathbf{x}' = A\mathbf{x} + \mathbf{F} \Rightarrow \Phi' \cdot \mathbf{v} + \Phi \cdot \mathbf{v}' = A\Phi \cdot \mathbf{v} + \mathbf{F}$.
Now substitute for $\Phi' = A\phi$:

$$\Rightarrow \quad A\Phi \cdot \mathbf{v} + \Phi \cdot \mathbf{v}' = A\Phi \cdot \mathbf{v} + \mathbf{F}.$$
$$\Rightarrow \quad \Phi \cdot \mathbf{v}' = \mathbf{F}$$
$$\Rightarrow \quad \mathbf{v}' = \Phi^{-1} \cdot \mathbf{F}$$
$$\Rightarrow \quad \mathbf{v} = \int \Phi^{-1} \cdot \mathbf{F} + \mathbf{C}.$$
$$\Rightarrow \quad \mathbf{x} = \Phi \cdot \mathbf{v} = \Phi \left(\int \Phi^{-1} \cdot \mathbf{F}\, dt + \mathbf{C} \right). \quad \text{QED.}$$

Definite integral version of variation of parameters

$$\mathbf{x}(t) = \Phi(t) \left(\int_{t_0}^{t} \Phi^{-1}(u) \cdot \mathbf{F}(u)\, du + \mathbf{C} \right), \quad \text{where} \quad \mathbf{C} = \Phi^{-1}(t_0) \cdot \mathbf{x}(t_0).$$

Example. Solve $\mathbf{x}' = \begin{pmatrix} 6 & 5 \\ 1 & 2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} e^t \\ e^{5t} \end{pmatrix}$

Notation: $A = \begin{pmatrix} 6 & 5 \\ 1 & 2 \end{pmatrix}$, $\mathbf{F} = \begin{pmatrix} 1 \\ t \end{pmatrix}$.

Fundamental matrix (earlier example): $\Phi = \begin{pmatrix} e^t & 5e^{7t} \\ -e^t & e^{7t} \end{pmatrix}$ $\Phi^{-1} = \frac{e^{-8t}}{6} \begin{pmatrix} e^{7t} & -5e^{7t} \\ e^t & e^t \end{pmatrix}$.

Variation of parameters: $\mathbf{x} = \Phi \int \Phi^{-1} \cdot \mathbf{F}\, dt$

$$\begin{aligned}
&= \Phi \int \frac{e^{-8t}}{6} \begin{pmatrix} e^{7t} & -5e^{7t} \\ e^{t} & e^{t} \end{pmatrix} \cdot \begin{pmatrix} e^{t} \\ e^{5t} \end{pmatrix} \quad = \quad \Phi \int \frac{1}{6} \begin{pmatrix} 1 - 5e^{4t} \\ e^{-6t} + e^{-2t} \end{pmatrix} dt \\
&= \frac{1}{6}\Phi \begin{pmatrix} t - \frac{5}{4}e^{4t} + c_1 \\ -\frac{1}{6}e^{-6t} - \frac{1}{2}e^{-2t} + c_2 \end{pmatrix} \quad = \quad \frac{1}{6} \begin{pmatrix} te^{t} - \frac{5}{4}e^{5t} - \frac{5}{6}e^{t} - \frac{5}{2}e^{5t} + c_1 e^{t} + 5c_2 e^{7t} \\ -te^{t} + \frac{5}{4}e^{5t} - \frac{1}{6}e^{t} - \frac{1}{2}e^{5t} - c_1 e^{t} + c_2 e^{7t} \end{pmatrix} \\
&= \frac{1}{6}\left[te^{t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{5t} \begin{pmatrix} -15/4 \\ 3/4 \end{pmatrix} + e^{t} \begin{pmatrix} -5/6 \\ -1/6 \end{pmatrix} + c_1 e^{t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{7t} \begin{pmatrix} 5 \\ 1 \end{pmatrix} \right].
\end{aligned}$$

(Notice the homogeneous solution appearing with the constants of integration).