Fundamental Matrices

In the literature, solutions to linear systems often are expressed using square matrices rather than vectors. This is an elegant bookkeeping technique and a very compact, efficient way to express these formulas. As before, we state the definitions and results for a 2×2 system, but they generalize immediately to $n \times n$ systems.

We return to the system

$$\mathbf{x}' = A(t)\,\mathbf{x}\,, \tag{1}$$

with the general solution

$$\mathbf{x} = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)\,, \tag{2}$$

where $\mathbf{x}_1$ and $\mathbf{x}_2$ are two independent solutions to (1), and c_1 and c_2 are arbitrary constants.

We form the matrix whose columns are the solutions $\mathbf{x}_1$ and $\mathbf{x}_2$:

$$\Phi(t) = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}. \tag{3}$$

Since the solutions are linearly independent, we called them a *fundamental* set of solutions, and therefore we call the matrix in (3) a **fundamental matrix** for the system (1).

Writing the general solution using $\Phi(t)$. As a first application of $\Phi(t)$, we can use it to write the general solution (2) efficiently. For according to (2), it is

$$\mathbf{x} = c_1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + c_2 \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix},$$

which becomes using the fundamental matrix

$$\mathbf{x} = \Phi(t)\,\mathbf{c} \qquad \text{where } \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \ (\textit{general solution to (1)}). \tag{4}$$

Note that the vector $\mathbf{c}$ must be written on the right, even though the c's are usually written on the left when they are the coefficients of the solutions $\mathbf{x}_i$.

Solving the IVP using $\Phi(t)$. We can now write down the solution to the IVP

$$\mathbf{x}' = A(t)\,\mathbf{x}, \qquad \mathbf{x}(t_0) = \mathbf{x}_0. \tag{5}$$

Starting from the general solution (4), we have to choose the **c** so that the initial condition in (6) is satisfied. Substituting t_0 into (5) gives us the matrix equation for **c** :

$$\Phi(t_0)\,\mathbf{c} = \mathbf{x}_0 .$$

Since the determinant $|\Phi(t_0)|$ is the value at t_0 of the Wronskian of $\mathbf{x}_1$ and $\mathbf{x}_2$, it is non-zero since the two solutions are linearly independent (Theorem 3 in the note on the Wronskian). Therefore the inverse matrix exists and the matrix equation above can be solved for **c**:

$$\mathbf{c} = \Phi(t_0)^{-1}\mathbf{x}_0.$$

Using the above value of **c** in (4), the solution to the IVP (1) can now be written

$$\mathbf{x} = \Phi(t)\Phi(t_0)^{-1}\mathbf{x}_0 . \tag{6}$$

Note that when the solution is written in this form, it's "obvious" that $\mathbf{x}(t_0) = \mathbf{x}_0$, i.e., that the initial condition in (5) is satisfied.

An equation for fundamental matrices We have been saying "a" rather than "the" fundamental matrix since the system (1) doesn't have a unique fundamental matrix: there are many ways to pick two independent solutions of $\mathbf{x}' = A\,\mathbf{x}$ to form the columns of Φ. It is therefore useful to have a way of recognizing a fundamental matrix when you see one. The following theorem is good for this; we'll need it shortly.

Theorem 1 *$\Phi(t)$ is a fundamental matrix for the system (1) if its determinant $|\Phi(t)|$ is non-zero and it satisfies the matrix equation*

$$\Phi' = A\,\Phi, \tag{7}$$

where Φ' means that each entry of Φ has been differentiated.

Proof. Since $|\Phi| \not\equiv 0$, its columns $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent, as we saw in the previous note. Let $\Phi = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix}$. According to the rules for matrix multiplication (7) becomes

$$\begin{pmatrix} \mathbf{x}_1' \\ \mathbf{x}_2' \end{pmatrix} = A \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} = \begin{pmatrix} A\mathbf{x}_1 \\ A\mathbf{x}_2 \end{pmatrix}.$$

which shows that

$$\mathbf{x}_1' = A\,x_1 \qquad \text{and} \qquad \mathbf{x}_2' = A\,\mathbf{x}_2\,;$$

this last line says that $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions to the system (1). $\square$