Existence and Uniqueness and Superposition in the General Case

We can extend the results above to the inhomogeneous case.

$$\mathbf{x}' = A(t)\mathbf{x} \text{ (homogeneous)} \tag{H}$$

$$\mathbf{x}' = A(t)\mathbf{x} + \mathbf{F}(t) \text{ (inhomogeneous),} \tag{I}$$

where $F(t)$ is the *input* to the system.

Linearity/superposition:

1. If $\mathbf{x_1}$ and $\mathbf{x_2}$ are solutions to (H) then so is $\mathbf{x} = c_1\mathbf{x_1} + c_2\mathbf{x_2}$

2. If $\mathbf{x_h}$ is a solution to (H) and $\mathbf{x_p}$ is a solution to (I) then $\mathbf{x} = \mathbf{x_h} + \mathbf{x_p}$ is also a solution to (I).

3. If $\mathbf{x_1}' = A\mathbf{x_1} + \mathbf{F_1}$, and $\mathbf{x_2}' = A\mathbf{x_2} + \mathbf{F_2}$ then $\mathbf{x} = \mathbf{x_1} + \mathbf{x_2}$ satisfies $\mathbf{x}' = A\mathbf{x} + \mathbf{F_1} + \mathbf{F_2}$. That is, superposition of inputs leads to superposition of outputs.

proof: 1. $\mathbf{x}' = c_1\mathbf{x_1}' + c_2\mathbf{x_2}' = c_1A\mathbf{x_1} + c_2A\mathbf{x_2} = A(c_1\mathbf{x_1} + c_2\mathbf{x_2}) = A\mathbf{x}$.

2. $\mathbf{x}' = \mathbf{x_h}' + \mathbf{x_p}' = A\mathbf{x_h} + A\mathbf{x_p} + \mathbf{F} = A(\mathbf{x_h} + \mathbf{x_p}) + \mathbf{F} = A\mathbf{x} + \mathbf{F}$.

3. $\mathbf{x}' = \mathbf{x_1}' + \mathbf{x_2}' = A\mathbf{x_1} + \mathbf{F_1} + A\mathbf{x_2} + \mathbf{F_2} = A(\mathbf{x_1} + \mathbf{x_2}) + \mathbf{F_1} + \mathbf{F_2} = A\mathbf{x} + \mathbf{F_1} + \mathbf{F_2}$.

Existence and uniqueness: We start with an initial time t_0 and the initial value problem:

$$\mathbf{x}' = A(t)\mathbf{x} + \mathbf{F}(t), \ \mathbf{x}(t_0) = \mathbf{x_0}. \tag{IVP}$$

Theorem: If $A(t)$ and $\mathbf{F}(t)$ are continuous then there exists a unique solution to (IVP).