Part II Problems and Solutions

Problem 1: *[Exponential matrix]*

(a) *We have seen that a complex number $z = a + bi$ determines a matrix $A(z)$ in the following way: $A(a+bi) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. This matrix represents the operation of multiplication by z, in the sense that if $z(x+yi) = v + wi$ then $A(z)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} v \\ w \end{bmatrix}$. What is $e^{A(z)t}$? What is $A(e^{zt})$?*

(b) *Say that a pair of solutions $x_1(t), x_2(t)$ of the equation $m\ddot{x} + b\dot{x} + kx = 0$ is* normalized at $t = 0$ *if:*

$$x_1(0) = 1, \quad \dot{x}_1(0) = 0$$
$$x_2(0) = 0, \quad \dot{x}_2(0) = 1$$

For example, find the normalized pair of solutions to $\ddot{x} + 2\dot{x} + 2x = 0$. Then find e^{At} where A is the companion matrix for the operator $D^2 + 2D + 2I$.

(c) *Suppose that $e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $e^{2t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ satisfy the equation $\dot{\mathbf{u}} = A\mathbf{u}$.*

(i) Find solutions $\mathbf{u_1}(t)$ and $\mathbf{u_2}(t)$ such that $\mathbf{u_1}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{u_2}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(ii) Find e^{At}.

(iii) Find A.

Solution: **(a)** With $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, $p_A(\lambda) = \lambda^2 - 2a\lambda + (a^2 + b^2) = (\lambda - a)^2 + b^2$, so the eigenvalues are $a \pm bi$. An eigenvector for $\lambda_1 = a + bi$ is given by $\mathbf{v_1}$ such that $\begin{bmatrix} -bi & -b \\ b & -bi \end{bmatrix}\mathbf{v_1} = \mathbf{0}$, and we can take $\mathbf{v_1} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$. The corresponding normal mode is $e^{(a+bi)t}\begin{bmatrix} 1 \\ -i \end{bmatrix}$. Its real and imaginary parts give linearly independent real solutions, $e^{at}\begin{bmatrix} \cos(bt) \\ \sin(bt) \end{bmatrix}$ and $e^{at}\begin{bmatrix} \sin(bt) \\ \cos(bt) \end{bmatrix}$. So a fundamental matrix is given by $\Phi(t) = e^{at}\begin{bmatrix} \cos(bt) & \sin(bt) \\ \sin(bt) & -\cos(bt) \end{bmatrix}$.
$\Phi(0) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\Phi(0)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, so $e^{At} = \Phi(t)\Phi(0)^{-1} = e^{at}\begin{bmatrix} \cos(bt) & -\sin(bt) \\ \sin(bt) & \cos(bt) \end{bmatrix}$.

$$A(e^{(a+bi)t}) = A(e^{at}(\cos(bt) + i\sin(bt))) = e^{at}\begin{bmatrix} \cos(bt) & -\sin(bt) \\ \sin(bt) & \cos(bt) \end{bmatrix} = e^{A(a+bi)t}.$$

(b) $s^2 + 2s + 2 = (s+1)^2 + 1$ so the roots of the characteristic polynomial are $-1 \pm i$. Basic solutions are given by $y_1 = e^{-t}\cos(t)$ and $y_2 = e^{-t}\sin(t)$. (I write y instead of x because the problem wrote x for the normalized solutions.) $y_1(0) = 1$, $\dot{y}_1(0) = -1$, $y_2(0) = 0$, $\dot{y}_2(0) = 1$. So $x_1 = y_1 + y_2$ and $x_2 = y_2$ form a normalized pair of solutions: $x_1(t) = e^{-t}(\cos t + \sin t)$, $x_2(t) = e^{-t}\sin t$.

The companion matrix is $A = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}$. Its characteristic polynomial is the same, $\lambda^2 + 2\lambda + 2$, so its eigenvalues are the same, $-1 \pm i$. An eigenvector for value $-1+i$ is given by $\mathbf{v_1}$ such that $\begin{bmatrix} 1-i & 1 \\ -2 & -1-i \end{bmatrix}\mathbf{v_1} = \mathbf{0}$. We can take $\mathbf{v_1} = \begin{bmatrix} 1 \\ -1+i \end{bmatrix}$. The corresponding normal mode is $e^{(-1+i)t}\begin{bmatrix} 1 \\ -1+i \end{bmatrix}$, which has real and imaginary parts $\mathbf{u_1} = e^{-t}\begin{bmatrix} \cos t \\ -\cos t - \sin t \end{bmatrix}$ and $\mathbf{u_2} = e^{-t}\begin{bmatrix} \sin t \\ -\sin t + \cos t \end{bmatrix}$. $\Phi(t) = [\mathbf{u_1} \quad \mathbf{u_2}]$ has $\Phi(0) = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$. $\Phi(0)^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, so $e^{At} = \Phi(t)\Phi(0)^{-1} = e^{-t}\begin{bmatrix} \cos t + \sin t & \sin t \\ -2\sin t & -\sin t + \cos t \end{bmatrix}$.
The top entries coincide with x_1 and x_2 computed above.

(c) (i) $\mathbf{u_1} = c_1 e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ so $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \mathbf{u_1}(0) = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ c_1 + 2c_2 \end{bmatrix}$. Thus $c_1 = 2$ and $c_2 = -1$: $\mathbf{u_1} = \begin{bmatrix} 2e^{3t} - e^{2t} \\ 2e^{3t} - 2e^{2t} \end{bmatrix}$. Start again for $\mathbf{u_2}$: $\mathbf{u_2} = c_1 e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ so $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \mathbf{u_2}(0) = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ c_1 + 2c_2 \end{bmatrix}$. Thus $c_1 = -1$ and $c_2 = 1$: $\mathbf{u_2} = \begin{bmatrix} -e^{3t} + e^{2t} \\ -e^{3t} + 2e^{2t} \end{bmatrix}$.

(ii) We have just computed the columns of the exponential matrix:
$e^{At} = \begin{bmatrix} 2e^{3t} - e^{2t} & -e^{3t} + e^{2t} \\ 2e^{3t} - 2e^{2t} & -e^{3t} + 2e^{2t} \end{bmatrix}$.

(iii) The matrix A has eigenvalues 3 and 2, with eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. The $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. The top entries give the equations $a + b = 3$ and $a + 2b = 2$, which imply $a = 4$, $b = -1$. The bottom entries give the equations $c + d = 3$, $c + 2d = 4$, which imply $c = 2$, $d = 1$. Thus $A = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$.