Part II Problems

Problem 1: [Exponential matrix]

(a) We have seen that a complex number $z = a + bi$ determines a matrix $A(z)$ in the following way: $A(a+bi) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. This matrix represents the operation of multiplication by z, in the sense that if $z(x+yi) = v + wi$ then $A(z)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} v \\ w \end{bmatrix}$. What is $e^{A(z)t}$? What is $A(e^{zt})$?

(b) Say that a pair of solutions $x_1(t), x_2(t)$ of the equation $m\ddot{x} + b\dot{x} + kx = 0$ is *normalized at* $t = 0$ if:

$$x_1(0) = 1, \quad \dot{x}_1(0) = 0$$
$$x_2(0) = 0, \quad \dot{x}_2(0) = 1$$

For example, find the normalized pair of solutions to $\ddot{x} + 2\dot{x} + 2x = 0$. Then find e^{At} where A is the companion matrix for the operator $D^2 + 2D + 2I$.

(c) Suppose that $e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $e^{2t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ satisfy the equation $\dot{\mathbf{u}} = A\mathbf{u}$.

(i) Find solutions $\mathbf{u_1}(t)$ and $\mathbf{u_2}(t)$ such that $\mathbf{u_1}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{u_2}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(ii) Find e^{At}.

(iii) Find A.