18.03SC Differential Equations, Fall 2011
Transcript – Linearization

PROFESSOR: Welcome to this presentation on linearization. So here we consider a prey-predator system, where we have x dot equals x, factorizing 3 minus x minus xy, and y dot equals y factorizing 1 minus y plus xy. So here we can see that x is the prey, because y is basically feeding on it. And y is the predator, because feeding on x gives a growth of the predator species, y. So here, because it's a prey-predator system, x and y are assumed to be positive.

So you're asked to interpret even further this system and to find the critical points, linearize and sketch the phase portrait and then discuss what your linearization tells you about the behavior of this system. So why don't you pause the video and take a few minutes to do that. I'll be right back.

Welcome back. So I helped you already for the first question. We interpreted that x was the prey and y was the predator. The other thing that you could see is that this term here is basic logistic growth. So if we didn't have the predator species around, this species x of prey would just grow and eventually reach a value that corresponds to the saturation level. And if we started with a lot of prey, then eventually they would die off and go back to that same carrying capacity value of prey in that environment.

Same thing here for the predator. We have a logistic growth, with different growth coefficients. And again, if we didn't have the prey around, we would just have a logistic dynamic for this species on its own. But with the prey, we have a growth of this species. So that basically ends the first part.

For the second part, we need to find the critical points. So how do we find the critical points? The critical points correspond to basically f and g equals to 0. So f equal to 0 equals to g. You see from f equals to 0 that we can have either x equals 0, if we factorize the x on both terms, and if x is not equal to 0, then we end up with y equals to 3 minus x.

For the second part, we can also here, if g equals to 0, factorize y. And we get either y equals to 0, or we have that x is equal to y minus 1, where I just bring that term to the other hand. So the critical point would be either of these two entries. It would be, for example, x equals to 0 and this entry, which would give us y equals to 1, or we would have y equals to 0, and then this entry would give us x equals to 3. And then the last combined case, where we have these two entries, which corresponds to 1 and 2, if you do it.

So let's look now at the stability of the linearization of the system around each one of these critical points. So what do we do to linearize the system? Very quickly, we need to compute the Jacobian around each critical point-- and we're just going to build a table after to do that-- where we basically have f, a derivative of f around x, derivative of g around x, derivative of f around y, derivative of g around y, all this evaluated at the critical point.

And this corresponds to basically linearizing our nonlinear system, because you see here that there are a lot of nonlinear systems, and studying the stability around the neighborhood of the critical point, like if it was basically linear around there. This method has its limitations, and we'll discuss them later.

So first I'm going to just here give you the results of computation that I did earlier, where basically you can repeat this computation. But I don't want to spend too much time with the algebra here. So we have four critical points. (0,0), (0,1), (3,0), and (1,2). So this is going to just replace the values for the Jacobian. So here you compute basically the derivative of f of x with respect to x.

And you evaluate this at then the value of (0,0). And so that would give you (3,0) (0,1). For this case, same thing. We have the expression for the Jacobian. We evaluate it at critical point (0,1). And that gives us (2,0) (1,-1). For this one similarly, we evaluate the Jacobian at (3,0). And we get this Jacobian value. And for this last critical point, we have this Jacobian value.

So now what's next? So the Jacobian basically gives us the expression around the critical point to look at the system, like if it was linear around there. So from this point, we're back to the linear methods we learned before. We need to compute the eigenvalues of the Jacobian around each critical point, and then determine the structure around each critical point of the structure of basically the phase portrait.

So what are the eigenvalues? The eigenvalues are (3,1), and you can compute that and verify for yourself. Plus or minus root of 7pi and the whole thing over 2. So here basically, we have two real eigenvalues, both positive. So we're going to have, basically, an instability, unstable node. And it's just basically the local stability around the critical point.

Here we would have this one positive, one negative. So this is a saddle, which would be unstable. The (-3,4) would give us another saddle. And these complex eigenvalues would basically give us a spiral with the real part going to 0, with the real part being negative. So a spiral that's stable and [? symbiotical ?].

So let's just do the diagram here. And I'll continue the discussion. So let's consider only the case where x and y are positive, because we're talking about populations. And let's place our critical points. So we have a first critical points here at (0,0). We have a second critical point here at (0,1). We have a third critical point at (3,0), and the last critical points at basically (1,2), or something around there.

So now, based on the information we have on this table, we have the eigenvalues. We could also compute the corresponding eigenvectors. And you can compute that. And I'm not going to get into the details here, but basically the values of the eigenvectors would be important to give you, for example, the direction of your spiral. But we will do that on the diagram as we go.

So here at the (0,0) point is an unstable node. So basically, the solutions are going away from this point in the x and y. This point, (0,1) is a saddle. So we basically are on the ray here. You would compute that the eigenvector that corresponds to this negative eigenvalue would actually be in the direction (0,1) and would converge toward this solution.

And you can compute that the other eigenvector corresponding to the eigenvalue 2 would have a direction this form. So here we neglect what's happening here, but it would be in this direction. And the solutions would be basically going away from here. And we would have locally something like that.

For this point, which corresponds to an unstable mode, basically the solution would be going away. For this point, which corresponds to (3,0), we have a saddle again, which is unstable. And here you can compute that the eigenvector corresponding to the negative eigenvalue would basically be parallel to the x-axis. So it would have coordinates (1,0). And the eigenvector corresponding to the eigenvalue 4 would be directed in this direction. And basically, the solution would be fleeing from there. So locally, we would have something like that, like we did before.

So we'll complete the graph. Now, let's focus on the last critical point, (1,2). So this point corresponds to a spiral. And it's a stable spiral -- asymptotically stable, because these eigenvalue's negative. So the solutions are going toward this point. And you can look at the lower entry here of the matrix, which is positive, which means that we would be going counterclockwise. So we would have something there would be looking like this, going toward this point.

So that's roughly what we would have for the three critical points. Am I missing one? For the four critical points.

And so now we can complete our diagram by basically linking to different localized phase portraits together. And so what will we have here? So for example, we would have a solution here that would be escaping, if we start in this neighborhood, would be escaping from this critical point. But then eventually, it would get attracted by this other critical point, which when it enters its basin of attraction, given that it's symptotically stable would be attracted by it and go here.

If we started from a very high y value, we would be going down and then eventually getting close to this critical point that would then basically cause this directory to, again, escape and go feed this spiral by reaching the basin of attraction of this point.

If we look at the critical point (3,0), then if we start with a population x that is very large and we approach this point, then we would have a solution that basically would eventually continue parallel to this ray of the unstable part of the critical point (3,0), follow this ray and eventually basically just be far more parallel to this trajectory that links this linear (3,0), critical point to the (1,2) critical point. And so we can complete the diagram by having something like this.

Now for this point, what do we have? So for this point, we also have the solutions that would be basically fleeting from the point (0,0), and eventually could be attracted to the spiral as well. So that we give us then something like this, a trajectory that would be looking like that And basically these trajectories would be looking like that. And I'm not going to complete the parts where y and x are not positive. So we can complete the phase diagram of this nonlinear system in this way.

So now, how do we interpret this? What does this mean? Well, if we remind ourselves of what this is actually modeling, and if we just look at, for example, the different axes, what does the y equals 0, x equals 0 point mean? It means that basically we have 0 population of prey, 0 of predator. And so it makes sense that we have basically an unstable point here, an unstable critical point, because as soon as we add one prey or one predator, we would have an increase of the prey population, eventually the predator would grow, and so we would have a solution that basically escapes the area around the critical point (0,0).

What would happen if we just looked at the axis y equals to 0? y equals to 0 corresponds to dynamics where we don't have the predator population. So the prey is just leaving its life, growing at logistic growth. And so basically, it's attracted by the carrying capacity that would be here set at 3. And so if we start with a lot of prey, eventually they die out until they reach population 3. And if we start with not enough, they grow and they reach population 3, without the predator around. Same thing for the predator on its own.

Now if we put the two together then we have a spiral, which means that we have oscillation. If we have a lot of predators, we have very few prey. And eventually the predators start dying off as well. But then, because they start dying off, the prey population starts increasing, which then gives an increase of the predator. So we get, eventually, an oscillation that goes to this attractor, (1,2), where the system will stabilize eventually, where we would have, basically, one prey for two predators.

So that ends the interpretation of the system. And the idea here was to basically use what we learned for the linear systems, in terms of the phase portrait, and to see how we can apply that to the nonlinear cases, after linearlizing the system around each one of the critical points.

What I should also mention here is that in all these cases that we looked at, we had cases that were structurally stable, which means that in our determinant trace diagram, we weren't at any borderline case where a little perturbation could make the structure around the critical point change radically. So all these points are structurally stable, and the linearization therefore is valid around them. And that ends this recitation.