Structural Stability for Non-linear Systems

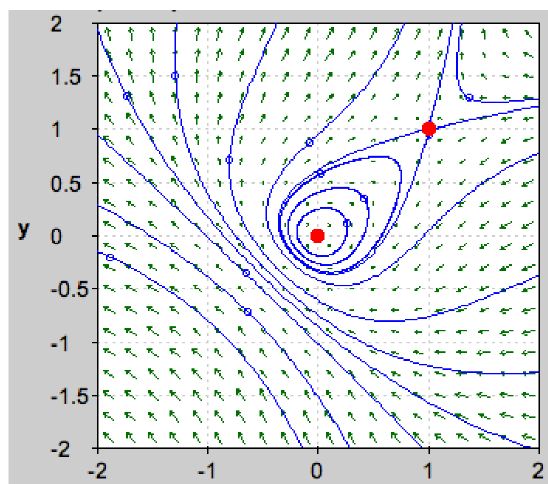
In the preceding note we discussed the structural stability of a linear system. How does it apply to non-linear systems?

Suppose our non-linear system has a critical point at P, and we want to study its trajectories near P by linearizing the system at P.

This linearization is only an approximation to the original system, so if it turns out to be a borderline case, i.e., one sensitive to the exact value of the coefficients, *the trajectories near P of the original system can look like any of the types obtainable by slightly changing the coefficients of the linearization*.

It could also look like a combination of types. For instance, if the linearized system had a critical line (i.e., one eigenvalue zero), the original system could have a sink node on one half of the critical line, and an unstable saddle on the other half. (This actually occurs.)

In other words, the method of linearization to analyze a non-linear system near a critical point doesn't fail entirely, but we don't end up with a definite picture of the non-linear system near P; we only get a list of possibilities. In general one has to rely on computation or more powerful analytic tools to get a clearer answer. The first thing to try is a computer picture of the non-linear system, which often will give the answer.



Example. $x' = y - x^2, \quad y' = -x + y^2$

Jacobian: $J(x,y) = \begin{pmatrix} -2x & 1 \\ -1 & 2y \end{pmatrix}$

Crititcal points: $y - x^2 = 0 \Rightarrow y = x^2$
$-x + y^2 = 0 \Rightarrow -x + x^4 = 0 \Rightarrow x = 0, 1.$
$\Rightarrow$ $(0,0)$ and $(1,1)$ are the critical points.

$J(1,1) = \begin{pmatrix} -2 & 1 \\ -1 & 2 \end{pmatrix}$:

characteristic equation: $\lambda^2 - 3 = 0 \Rightarrow \lambda = \pm\sqrt{3} \Rightarrow$ linearized system has a saddle.

This is *structurally stable* $\Rightarrow$ the nonlinear system has a saddle at $(1,1)$.

$J(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$: eigenvalues = $\pm i \Rightarrow$ a linearized center.

This is *not structurally stable*. The nonlinear system could be any one of a center, spiral out or spiral in. Using a computer program it appears that (0,0) is in fact a center. (This can be proven using more advanced methods.)

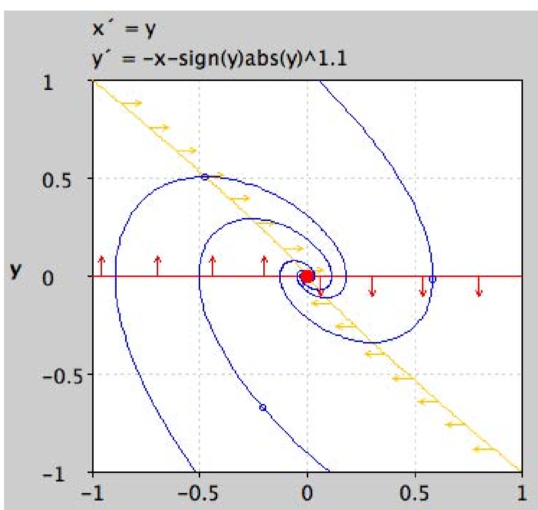
We can show the trajectories near (0,0) are not spirals by exploiting the symmetry of the picture. First note, if $(x(t), y(t)$ is a solution then so is $(y(-t), x(-t)$. That is, the trajectory is symmetric in the line $x = y$. This implies it can't be a spiral. Since the only other choice choice is that the critical point (0,0) is a center, the trajectories must be closed.

The following two examples show that a linearized center might also be a spiral in or a spiral out in the nonlinear system.

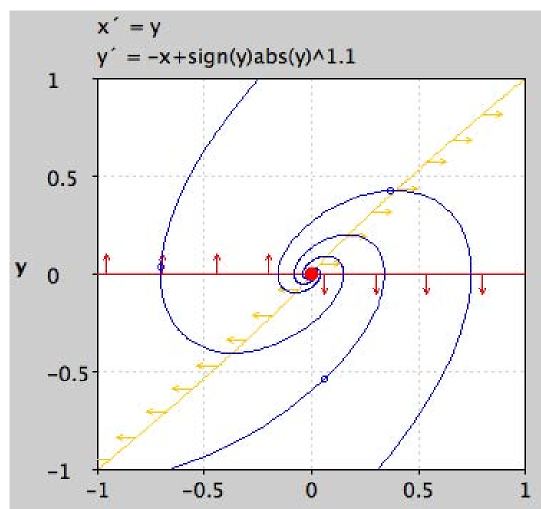
Example a. $x' = y,\ y' = -x - y^3$.
Example b. $x' = y,\ y' = -x + y^3$.
In both examples the only critical point is $(0, 0)$.
$J(0, 0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Rightarrow$ linearized center. This is not structurally stable.

In example a the critical point turns out to be a spiral sink. In example b it is a spiral source.

Below are computer-generated pictures. Because the y^3 term causes the spiral to have a lot of turns we 'improved' the pictures by using the power 1.1 instead.



Spiral in



Spiral out