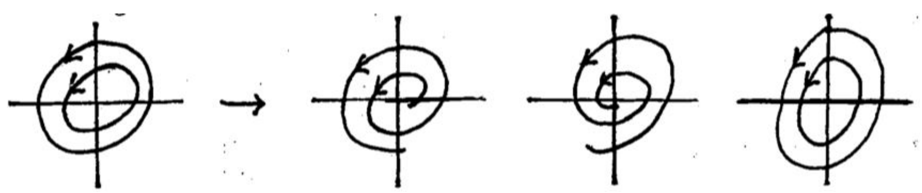
The Borderline Geometric Types

All the other possibilities for the linear system (**??**) we call borderline types. We will show now that none of them is structurally stable; we begin with the center.

Eigenvalues pure imaginary. Once again we use the eigenvalue with the positive imaginary part: $\lambda = 0 + si$, $s > 0$. It corresponds to a *center*: the trajectories are a family of concentric ellipses, centered at the origin. If the coefficients a, b, c, d are changed a little, the eigenvalue $0 + si$ changes a little to $r' + s'i$, where $r' \approx 0, s' \approx s$, and there are three possibilities for the new eigenvalue:

$0 + si \ \rightarrow \ r' + s'i$:	$r' > 0$	$r' < 0$	$r' = 0$
$s > 0$	$s' > 0$	$s' > 0$	$s' > 0$
center	*source spiral*	*sink spiral*	*center*

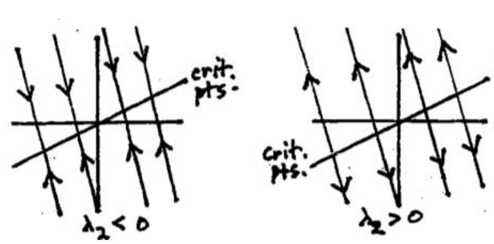
Correspondingly, there are three possibilities for how the geometric picture of the trajectories can change:



Eigenvalues real; one eigenvalue zero. Here $\lambda_1 = 0$, and $\lambda_2 > 0$ or $\lambda_2 < 0$. The general solution to the system has the form (α_1, α_2 are the eigenvectors)

$$\mathbf{x} \ = \ c_1\alpha_1 + c_2\alpha_2 e^{\lambda_2 t}.$$

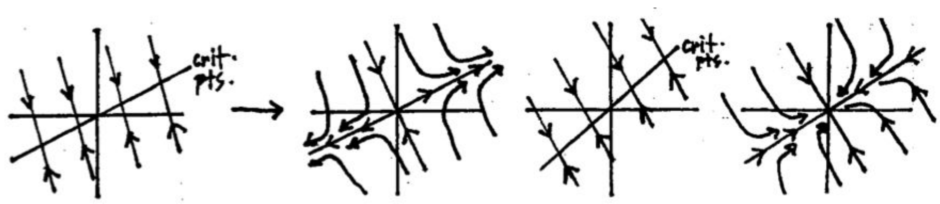
If $\lambda_2 < 0$, the geometric picture of its trajectories shows a line of critical points (constant solutions, corresponding to $c_2 = 0$), with all other trajectories being parallel lines ending up (for $t = \infty$) at one of the critical points, as shown below.



We continue to assume $\lambda_2 < 0$. As the coefficients of the system change a little, the two eigenvalues change a little also; there are three possibilities, since the eigenvalue $\lambda = 0$ can become positive, negative, or stay zero:

$\lambda_1 = 0 \;\rightarrow\; \lambda_1'$:	$\lambda_1' > 0$	$\lambda_1' = 0$	$\lambda_1 < 0$
$\lambda_2 < 0 \;\rightarrow\; \lambda_2'$:	$\lambda_2' < 0$	$\lambda_2' < 0$	$\lambda_2' < 0$
critical line	*unstable saddle*	*critical line*	*sink node*

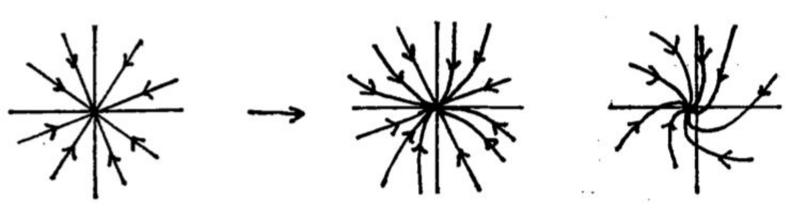
Here are the corresponding pictures. (The pictures would look the same if we assumed $\lambda_2 > 0$, but the arrows on the trajectories would be reversed.)



One repeated real eigenvalue. Finally, we consider the case where $\lambda_1 = \lambda_2$. Here there are a number of possibilities, depending on whether λ_1 is positive or negative, and whether the repeated eigenvalue is complete (i.e., has two independent eigenvectors), or defective (i.e., incomplete: only one eigenvector). Let us assume that $\lambda_1 < 0$. We vary the coefficients of the system a little. By the same reasoning as before, the eigenvalues change a little, and by the same reasoning as before, we get as the main possibilities (omitting this time the one where the changed eigenvalue is still repeated):

$\lambda_1 < 0$	$\rightarrow$	$\lambda_1' < 0$	$r + si$
$\lambda_2 < 0$	$\rightarrow$	$\lambda_2' < 0$	$r - si$
$\lambda_1 = \lambda_2$		$\lambda_1' \neq \lambda_2'$	$r \approx \lambda_1, s \approx 0,$
sink node		*sink node*	*sink spiral*

Typical corresponding pictures for the complete case and the defective (incomplete) case are (the last one is left for you to experiment with on the computer screen)



complete: star node *incomplete: defective node*

Remarks. Each of these three cases—one eigenvalue zero, pure imaginary eigenvalues, repeated real eigenvalue—has to be looked on as a borderline linear system: altering the coefficients slightly can give it an entirely different geometric type, and in the first two cases, possibly alter its stability as well.

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