18.03SC Practice Problems 32

First order linear systems

Vocabulary/Concepts: system of differential equations; linear, time-independent, homogeneous; matrix, matrix multiplication; solution, trajectory, phase portrait; companion matrix.

1. Practice in matrix multiplication: Compute the following products.

(a) $\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$, **(b)** $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} x & y \end{bmatrix}$, **(c)** $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$, **(d)** $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x & u \\ y & v \end{bmatrix}$.

2. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be any 2×2 matrix.

Multiplying by the matrix A sends any vector $\begin{bmatrix} x \\ y \end{bmatrix}$ to another vector $A \begin{bmatrix} x \\ y \end{bmatrix}$. This operation can be visualized by thinking about where it sends the square with corners

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ \mathbf{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ \mathbf{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ \mathbf{i}+\mathbf{j} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

For each of the following matrices A, draw segments connecting the dots $\mathbf{0}$, $A\mathbf{i}$, $A(\mathbf{i}+\mathbf{j})$, $A\mathbf{j}$, $\mathbf{0}$, and come up with a verbal description of the operation.

(a) $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

(d)$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(e) $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

3. Examine the equation

$$\ddot{x} + 2\dot{x} + 2x = 0.$$

(a) What is the companion matrix A of this second order equation?

(b) Find two independent real solutions of this equation.

(c) Now let $x_1(t)$ denote the solution with initial condition $x_1(0) = 0$, $\dot{x}_1(0) = 1$. Find it, and then write down the corresponding solution $\mathbf{u_1}(t) = \begin{bmatrix} x_1(t) \\ \dot{x}_1(t) \end{bmatrix}$ of the equation $\dot{\mathbf{u}} = A\mathbf{u}$. What is $\mathbf{u_1}(0)$?

Sketch the graphs of $x_1(t)$ and of $\dot{x}_1(t)$, and sketch the trajectory of the solution $\mathbf{u_1}(t)$. Compare these pictures.

(d) Sketch a few more trajectories to fill out the phase portrait. In particular, sketch the trajectory of $\mathbf{u_2}(t)$ with $\mathbf{u_2}(0) = \mathbf{i}$.

When trajectories of this companion equation cross the x axis, at what angle do they cross it?

4. Let $a + bi$ be a complex number. There is a matrix A such that if $(a + bi)(x + yi) = (v + wi)$ then

$$A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} v \\ w \end{bmatrix}.$$

(a) Find this matrix A for general $a + bi$.

(b) What is the matrix for $a + bi = 2$? For $a + bi = i$? For $a + bi = 1 + i$? Draw the parallelograms discussed in **(2)** for these matrices.

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