Part I Problems and Solutions

Problem 1: *Compute the following matrix products:*

a) $\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

b) $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} x & y \end{bmatrix}$

c) $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

d) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x & u \\ y & v \end{bmatrix}$

Solution:

a) $\begin{bmatrix} x + 2y \end{bmatrix}$

b) $\begin{bmatrix} x & y \\ 2x & 2y \end{bmatrix}$

c) $\begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$

d) $\begin{bmatrix} x + 2y & u + 2v \\ 3x + 4y & 3u + 4v \end{bmatrix}$

Problem 2: *Let* $A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$ *and* $B = \begin{bmatrix} 0 & -1 \\ 2 & 1 \end{bmatrix}$. *Show that* $AB \neq BA$.

Solution:

$$AB = \begin{bmatrix} 4 & 1 \\ -2 & -4 \end{bmatrix}$$

$$BA = \begin{bmatrix} -3 & 1 \\ 5 & 3 \end{bmatrix}$$

Problem 3: *Write the following equations as equivalent first-order systems.*

a) $\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + tx^2 = 0$

b) $y'' - x^2y' + (1 - x^2)y = \sin x$

Solution:

a) $x'' + 5x' + tx^2 = 0 \rightarrow\ x' = y,\ y' = -tx^2 - 5y$

b) $y'' - x^2y' + (1 - x^2)y = \sin x \rightarrow\ y' = z,\ z' = (x^2 - 1)y + x^2z + \sin x$

Problem 4: *Solve the system* $x' = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x$ *in two ways:*

a) Solve the second equation, substitute for y *in the first equation, and solve it.*

b) Eliminate y *by solving the first equation for* y*, then substitute into the second equation, getting a second order equation for* x*. Solve it, and then find* y *from the first equation. Do your two methods give the same answer?*

Solution:

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

or $x' = x + y, y' = y$.

a) From the second equation, $y = c_1e^t$, so $x' - x = c_1e^t$, so the solution is $x = c_2e^t + c_1te^t$, $y = c_1e^t$.

b) Here we eliminate y instead. $y = x' - x$ so $(x' - x)' = x' - x \rightarrow x'' - 2x' + x = 0 \rightarrow$ $(m - 1)^2 = 0$ (char. eqn.). Thus, we have $x = c_1e^t + c_2te^t$, $y = c_2e^t$ (since $y = x' - x$). This is the same as before, with c_1, c_2 switched.