Part II Problems and Solutions

Problem 1: *[Linear systems and matrices]* **(a)** *We'll work with the two homogeneous constant coefficient linear equations* $\ddot{x} + 4\dot{x} + 3x = 0$ *and* $\ddot{x} + \dot{x} + \frac{5}{2}x = 0$*. For each, find two linearly independent real solutions (Please use either exponentials or functions of the form* $e^{rt}\cos(\omega t)$ *or* $e^{rt}\sin(\omega t)$*, and denote them by* $x_1(t)$ *and* $x_2(t)$*.), write down the general real solution, and determine the damping type. Also compute* $\dot{x}_1$ *and* $\dot{x}_2$*.*

(b) *Now write down the companion matrix for each of these two equations.*

(c) *Open the Mathlet* `Linear Phase Portraits: Matrix Entry`*. Select "Companion Matrix," and set the* c *and* d *values to the entries of the companion matrix for the first equation. (Note that clicking on a hashmark on a slider sets the value. In some cases you will have to accept the nearest possible value to the desired value.)*

For a companion matrix $A = \begin{pmatrix} 0 & 1 \\ c & d \end{pmatrix}$*, the colorful window at the upper left shows* $(d, -c)$*. You can adjust them by a cursor movement over that plane.*

The big window shows the "phase plane" of the system. It displays the trajectories of a few solutions. (The trajectories are the solution curves $(x(t), y(t))$ *in the* xy *plane, with the arrows indicating the direction of increasing* t*. We will study the phase plane trajectory portraits in detail in a later session.)*
Click on the window to produce more. You can clear them all using `[Clear]`*, and return to the original set of trajectories by re-setting one of the* c *or* d *sliders. Do this; return to the originally displayed selection of trajectories.*

Since $y = \dot{x}$*, a solution to* $\dot{\mathbf{u}} = A\mathbf{u}$ *is given by* $\begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}$ *where* $x(t)$ *is a solution of* $\ddot{x} + 4\dot{x} + 3x = 0$ *(in this first case). Draw a picture of the phase plane. Each of these trajectory curves should have an arrow on it indicating the direction of time: please indicate this on your picture. Identify which of the trajectories correspond to each of the basic solutions you found in* **(a)***. (These will be among the originally chosen trajectories.)*

(d) *There are some hook-shaped trajectories. The picture doesn't show a scale; but suppose that the largest displayed value of* x *is* $x = 2$*. A trajectory crosses the* x *axis at* $x = 1$*. What is the solution having this as its trajectory assuming that this crossing occurs at* $t = 0$*? Sketch the graph of the corresponding solution* $x(t)$ *of the original second order equation.*

(e) *Write down the general solution having the same trajectory. (Hint: suppose the solution has* $x(a) = 1$*).*

(f) *Now set the* c *and* d *sliders to the values relevant to the second equation you solved in* **(a)***. Sketch the phase portrait (and include the arrows indicating the direction of time).*

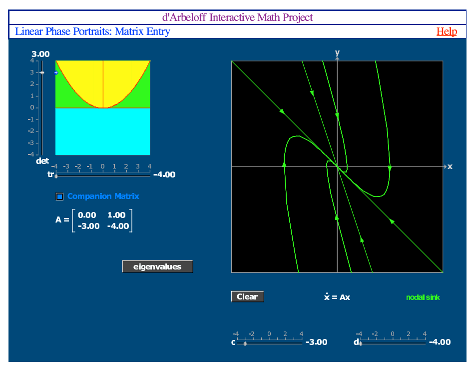
Using the same scale as above, what is the solution which passes through $(0,1)$ *at* $t=0$*? At what other times does this solution cross the* y *axis? Sketch, roughly, the graphs of* $x(t)$ *and of* $y(t)$*.*

(g) *In these companion matrix examples, whenever a trajectory crosses the* x *axis it seems to do it perpendicularly: its tangent vector is vertical. Explain.*

Solution: **(a)** $p(s) = s^2+4s+3 = (s+2)^2-1$ has roots $r_1=-1$ and $r_2=-3$. Basic solutions are given by $x_1 = e^{-t}$ and $x_2 = e^{-3t}$. (The order is not determined, and in fact any other pair of linearly independent solutions count as "basic.") $\dot{x}_1 = -e^{-t}$, $\dot{x}_2 = -3e^{-3t}$.

$p(s) = s^2+s+\frac{5}{2} = (s+\frac{1}{2})^2+\frac{9}{4}$ has roots $r = -\frac{1}{2} \pm \frac{3}{2}i$. Basic solutions are given by $x_1 = e^{-t/2}\cos(\frac{3}{2}t)$ and $x_2 = e^{-t/2}\sin(\frac{3}{2}t)$. (Same caveats as above.) $\dot{x}_1 = e^{-t/2}(-\frac{1}{2}\cos(\frac{3}{2}t) - \frac{3}{2}\sin(\frac{3}{2}t))$. $\dot{x}_2 = e^{-t/2}(-\frac{1}{2}\sin(\frac{3}{2}t)+\frac{3}{2}\cos(\frac{3}{2}t))$.

(b) $\begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ -\frac{5}{2} & -1 \end{bmatrix}$.

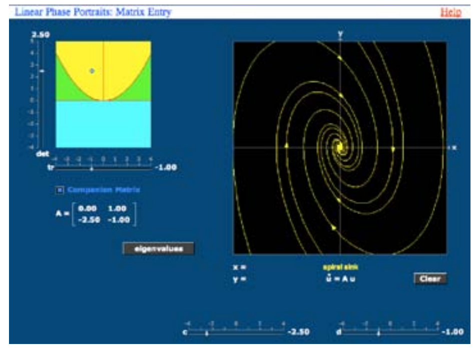


(c) The ray containing $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ corresponds to x_1; the ray containing $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ corresponds to x_2.

(d) The solution passing through $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ at $t=0$ is $\begin{bmatrix} x \\ \dot{x} \end{bmatrix}$ where x is the solution to $\ddot{x}+4\dot{x}+3x=0$ with $x(0)=1$, $\dot{x}(0)=0$. The general solution $x(t) = c_1e^{-t}+c_2e^{-3t}$ has $x(0)=c_1+c_2$ and $\dot{x}(0) = -c_1-3c_2$, so $c_1+c_2=1$ and $-c_1-3c_2=0$. Thus $c_2 = -\frac{1}{2}$ and $c_1 = \frac{3}{2}$ and $x(t) = \frac{1}{2}(3e^{-t}-e^{-3t})$, so $\mathbf{u}(t) = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 3e^{-t}-e^{-3t} \\ -3e^{-t}+3e^{-3t} \end{bmatrix}$. Description of the graph of $x(t)$: For $t << 0$ it is very negative and increasing. $x(t)=0$ for $t = -\frac{\ln 3}{2}$. It reaches a maximum $x(0)=1$, and then falls back through an inflection point to become asymptotic to $x=0$ as $t \to \infty$.

(e) $\mathbf{u}(t) = \frac{1}{2}\begin{bmatrix} 3e^{-(t-a)} - e^{-3(t-a)} \\ -3e^{-(t-a)} + 3e^{-3(t-a)} \end{bmatrix}$.

(f) The trajectory of interest is the spiral passing through $(0, 1)$.



The solution $\mathbf{u}(t) = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}$ passing through $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ at $t = 0$ is given by the solution $x(t)$ of $\ddot{x} + \dot{x} + \frac{5}{2}x = 0$ with $x(0) = 0$, $\dot{x}(0) = 1$. The general solution $x(t) = e^{-t/2}(c_1 \cos(\frac{3}{2}t) + c_2 \sin(\frac{3}{2}t))$ has $x(0) = c_1$ and $\dot{x}(0) = -\frac{1}{2}c_1 + \frac{3}{2}c_2$, so $c_1 = 0$ and $c_2 = \frac{2}{3}$. Thus $x(t) = \frac{2}{3}e^{-t/2}\sin(\frac{3}{2}t)$, $\dot{x}(t) = e^{-t/2}(\cos(\frac{3}{2}t) - \frac{1}{3}\sin(\frac{3}{2}t))$, and $\mathbf{u}(t) = e^{-t/2}\begin{bmatrix} \frac{2}{3}\sin(\frac{3}{2}t) \\ \cos(\frac{3}{2}t) - \frac{1}{3}\sin(\frac{3}{2}t) \end{bmatrix}$. This passes through the y axis when $x(t) = 0$, i.e. when $\sin(\frac{3}{2}t) = 0$, which is when t is an integral multiple of $2\pi/3$.

Both $x(t)$ and $y(t)$ are damped sinuosoids, and $x(0) = 0$ and $\dot{x}(0) = 1$, $y(0) = 1$, $\dot{y}(0) = -1$.

(g) The velocity vector of $\mathbf{u}(t) = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}$ is $\dot{\mathbf{u}}(t) = \begin{bmatrix} \dot{x}(t) \\ \ddot{x}(t) \end{bmatrix}$. When $\dot{x}(t) = 0$, then, $\dot{\mathbf{u}}(t) = \begin{bmatrix} 0 \\ \ddot{x}(t) \end{bmatrix}$, which is vertical. Alternatively, $\dot{\mathbf{u}} = A\mathbf{u}$ and $A = \begin{bmatrix} 0 & 1 \\ c & d \end{bmatrix}$, so if $\mathbf{u} = \begin{bmatrix} x \\ 0 \end{bmatrix}$ then $\dot{\mathbf{u}} = \begin{bmatrix} 0 & 1 \\ c & d \end{bmatrix}\begin{bmatrix} x \\ 0 \end{bmatrix} = x\begin{bmatrix} 0 \\ c \end{bmatrix}$, which is vertical.