Part II Problems

Problem 1: [Linear systems and matrices] **(a)** We'll work with the two homogeneous constant coefficient linear equations $\ddot{x} + 4\dot{x} + 3x = 0$ and $\ddot{x} + \dot{x} + \frac{5}{2}x = 0$. For each, find two linearly independent real solutions (Please use either exponentials or functions of the form $e^{rt}\cos(\omega t)$ or $e^{rt}\sin(\omega t)$, and denote them by $x_1(t)$ and $x_2(t)$.), write down the general real solution, and determine the damping type. Also compute $\dot{x}_1$ and $\dot{x}_2$.

(b) Now write down the companion matrix for each of these two equations.

(c) Open the Mathlet `Linear Phase Portraits: Matrix Entry`. Select "Companion Matrix," and set the c and d values to the entries of the companion matrix for the first equation. (Note that clicking on a hashmark on a slider sets the value. In some cases you will have to accept the nearest possible value to the desired value.)

For a companion matrix $A = \begin{pmatrix} 0 & 1 \\ c & d \end{pmatrix}$, the colorful window at the upper left shows $(d, -c)$. You can adjust them by a cursor movement over that plane.

The big window shows the "phase plane" of the system. It displays the trajectories of a few solutions. (The trajectories are the solution curves $(x(t), y(t))$ in the xy plane, with the arrows indicating the direction of increasing t. We will study the phase plane trajectory portraits in detail in a later session.)
Click on the window to produce more. You can clear them all using `[Clear]`, and return to the original set of trajectories by re-setting one of the c or d sliders. Do this; return to the originally displayed selection of trajectories.

Since $y = \dot{x}$, a solution to $\dot{\mathbf{u}} = A\mathbf{u}$ is given by $\begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}$ where $x(t)$ is a solution of $\ddot{x} + 4\dot{x} + 3x = 0$ (in this first case). Draw a picture of the phase plane. Each of these trajectory curves should have an arrow on it indicating the direction of time: please indicate this on your picture. Identify which of the trajectories correspond to each of the basic solutions you found in **(a)**. (These will be among the originally chosen trajectories.)

(d) There are some hook-shaped trajectories. The picture doesn't show a scale; but suppose that the largest displayed value of x is $x = 2$. A trajectory crosses the x axis at $x = 1$. What is the solution having this as its trajectory assuming that this crossing occurs at $t = 0$? Sketch the graph of the corresponding solution $x(t)$ of the original second order equation.

(e) Write down the general solution having the same trajectory. (Hint: suppose the solution has $x(a) = 1$).

(f) Now set the c and d sliders to the values relevant to the second equation you solved in **(a)**. Sketch the phase portrait (and include the arrows indicating the direction of time).

Using the same scale as above, what is the solution which passes through $(0, 1)$ at $t = 0$? At what other times does this solution cross the y axis? Sketch, roughly, the graphs of $x(t)$ and of $y(t)$.

(g) In these companion matrix examples, whenever a trajectory crosses the x axis it seems to do it perpendicularly: its tangent vector is vertical. Explain.