The Van der Pol Equation

An important kind of second-order non-linear autonomous equation has the form

$$x'' + u(x)\,x' + v(x) = 0 \qquad \textit{(Liénard equation)}\,. \tag{1}$$

One might think of this as a model for a spring-mass system where the damping force $u(x)$ depends on position (for example, the mass might be moving through a viscous medium of varying density), and the spring constant $v(x)$ depends on how much the spring is stretched—this last is true of all springs, to some extent. We also allow for the possibility that $u(x) < 0$ (i.e., that there is "negative damping").

The system equivalent to (1) is

$$\begin{aligned} x' &= y \\ y' &= -v(x) - u(x)\,y \end{aligned} \tag{2}$$

Under certain conditions, the system (2) has a unique stable limit cycle, or what is the same thing, the equation (1) has a unique periodic solution; and all nearby solutions tend towards this periodic solution as $t \to \infty$. The conditions which guarantee this were given by Liénard, and generalized in the following theorem.

Levinson-Smith Theorem *Suppose the following conditions are satisfied.*

(a) $u(x)$ *is even and continuous,*

(b) $v(x)$ *is odd,* $v(x) > 0$ *if* $x > 0$, *and* $v(x)$ *is continuous for all* x,

(c) $V(x) \to \infty$ *as* $x \to \infty$, *where* $V(x) = \int_0^x v(t)\,dt$,

(d) *for some* $k > 0$, *we have*

$$\left.\begin{array}{ll} U(x) < 0, & \text{for } 0 < x < k, \\ U(x) > 0 \text{ and increasing}, & \text{for } x > k, \\ U(x) \to \infty, & \text{as } x \to \infty, \end{array}\right\} \qquad \text{where} \quad U(x) = \int_0^x u(t)\,dt.$$

Then, the system (2) has

i) a unique critical point at the origin;

ii) a unique non-zero closed trajectory C, which is a stable limit cycle around the origin;

iii) all other non-zero trajectories spiralling towards C as $t \to \infty$.

We omit the proof, as too difficult. A classic application is to the equation

$$x'' - a(1 - x^2)\, x' + x = 0 \qquad \textit{(van der Pol equation)} \qquad (3)$$

which describes the current $x(t)$ in a certain type of vacuum tube. (The constant a is a positive parameter depending on the tube constants.) The equation has a unique non-zero periodic solution. Intuitively, think of it as modeling a non-linear spring-mass system. When $|x|$ is large, the restoring and damping forces are large, so that $|x|$ should decrease with time. But when $|x|$ gets small, the damping becomes negative, which should make $|x|$ tend to increase with time. Thus it is plausible that the solutions should oscillate; that it has exactly one periodic solution is a more subtle fact.

There is a lot of interest in limit cycles, because of their appearance in systems which model processes exhibiting periodicity. But not a great deal is known about them – this is still an area of active research.

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