Showing Limit Cycles Exist

The main tool which historically has been used to show that the system

$$\begin{aligned} x' &= f(x,y) \\ y' &= g(x,y) \end{aligned} \tag{1}$$

has a stable limit cycle is the

Poincare-Bendixson Theorem *Suppose R is the finite region of the plane lying between two simple closed curves* D_1 *and* D_2*, and* ***F*** *is the velocity vector field for the system (1). If*

(i) at each point of D_1 *and* D_2*, the field* ***F*** *points toward the interior of R, and*

(ii) R contains no critical points,

then the system (1) has a closed trajectory lying inside R.

The hypotheses of the theorem are illustrated by fig. 1. We will not give the proof of the theorem, which requires a background in Mathematical Analysis. Fortunately, the theorem strongly appeals to intuition. If we start on one of the boundary curves, the solution will enter R, since the velocity vector points into the interior of R. As time goes on, the solution can never leave R, since as it approaches a boundary curve, trying to escape from R, the velocity vectors are always pointing inwards, forcing it to stay inside R. Since the solution can never leave R, the only thing it can do as $t \to \infty$ is either approach a critical point — but there are none, by hypothesis — or spiral in towards a closed trajectory. Thus there is a closed trajectory inside R. (It cannot be an unstable limit cycle—it must be one of the other three cases shown above.)

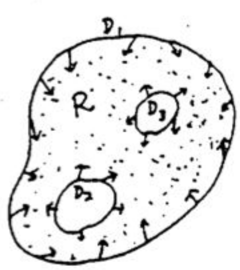

fig. 1

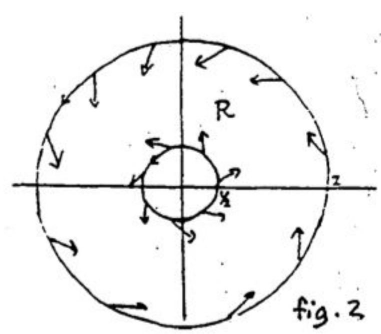

fig. 2

To use the Poincare-Bendixson theorem, one has to search the vector field for closed curves D along which the velocity vectors all point towards the same side. Here is an example where they can be found.

Example 1. Consider the system

$$\begin{aligned} x' &= -y + x(1 - x^2 - y^2) \\ y' &= x + y(1 - x^2 - y^2) \end{aligned} \tag{2}$$

Figure 2 shows how the associated velocity vector field looks on two circles. On a circle of radius 2 centered at the origin, the vector field points inwards, while on a circle of radius 1/2, the vector field points outwards. To prove this, we write the vector field along a circle of radius r as

$$\mathbf{x}' = (-y\mathbf{i} + x\mathbf{j}) + (1 - r^2)(x\mathbf{i} + y\mathbf{j}) \ . \tag{3}$$

The first vector on the right side of (3) is tangent to the circle; the second vector points radially *in* for the big circle ($r = 2$), and radially *out* for the small circle ($r = 1/2$). Thus the sum of the two vectors given in (3) points inwards along the big circle and outwards along the small one.

We would like to conclude that the Poincare-Bendixson theorem applies to the ring-shaped region between the two circles. However, for this we must verify that R contains no critical points of the system. We leave you to show as an exercise that $(0, 0)$ is the only critical point of the system; this shows that the ring-shaped region contains no critical points.

The above argument shows that the Poincare-Bendixson theorem can be applied to R, and we conclude that R contains a closed trajectory. In fact, it is easy to verify that $x = \cos t$, $y = \sin t$ solves the system, so the unit circle is the locus of a closed trajectory. We leave as another exercise to show that it is actually a stable limit cycle for the system, and the only closed trajectory.