Higher Order Unit Impulse Response

We can extend our reasoning in the first and second order cases to any order. Consider an n^{th} order system with DE

$$a_n x^{(n)} + a_{n-1} x^{(n-1)} + \ldots + a_1 x' + a_0 x = f(t)\,, \qquad (1)$$

where we take $f(t)$ to be the input. The equation for the unit impulse response of this system is

$$a_n x^{(n)} + a_{n-1} x^{(n-1)} + \ldots + a_1 x' + a_0 x = \delta(t), \quad \text{with rest IC.} \qquad (2)$$

The effect of the δ function input is to cause a jump in the $n-1^{\text{st}}$ derivative at time $t = 0$, while the lower order derivatives do not jump. That is, the system is put in the state

$$x(0^+) = 0,\ x'(0^+) = 0, \ldots, x^{(n-2)}(0^+) = 0,\ x^{(n-1)}(0^+) = 1/a_n.$$

To show this we use the same reasoning as in the second order case. Suppose there was a jump in a lower derivative. For example, suppose

$$x^{(n-3)}(0^+) = b \neq 0.$$

Then the expression for $x^{(n-2)}(t)$ contains $b\delta(t)$, which implies that $x^{(n-1)}(t)$ contains $b\delta'(t)$ and $x^{(n)}(t)$ contains $b\delta''(t)$. This is impossible because the right-hand side of (2) does not have any derivatives of the delta function.

Since $x^{n-1}(t)$ has a jump of $x^{(n-1)}(0^+) = 1/a_n$ at $t = 0$, its derivative $a_n x^{(n)}(t)$ has a unit impulse, $\delta(t)$, at $t = 0$.

We conclude that the solution to (2) is 0 for $t < 0$ and for $t > 0$ it is exactly the same as the solution to

$$a_n x^{(n)} + a_{n-1} x^{(n-1)} + \ldots + a_1 x' + a_0 x = 0$$

with initial conditions

$$x(0) = 0,\ x'(0) = 0, \ldots, x^{n-2}(0) = 0,\ x^{n-1}(0) = 1/a_n.$$