# Second order Unit Impulse Response

## 1. Effect of a Unit Impulse on a Second order System

We consider a second order system

$$m\ddot{x} + b\dot{x} + kx = f(t). \tag{1}$$

Our first task is to derive the following. If the input $f(t)$ is an impulse $c\delta(t-a)$, then the system's response to $f(t)$ has the following properties.
1. The momentum $m\dot{x}(t)$ jumps by $c$ units at $t = a$. That is,

$$m\dot{x}(a^+) - m\dot{x}(a^-) = c.$$

2. The position $x(t)$ is unchanged at $t = a$. That is,

$$x(a^+) = x(a^-).$$

Recall the argument that we used before: If $x(t)$ had a jump at $a$ then $\dot{x}(t)$ would contain a multiple of $\delta(t-a)$. So, $m\ddot{x}(t)$ would contain a multiple of the doublet $\delta'(t-a)$. This is impossible since the input $\delta(t-a)$ does not contain a doublet. This shows point (2) above.

To show point (1), we note that if $m\dot{x}(t)$ has a jump of $c$ units at $t-a$ then $m\ddot{x}(t)$ contains the term $c\delta(t-a)$. This is needed to make the left-hand side of equation (1) match the right hand side when $f(t) = c\delta(t-a)$.

Another way to show points (1) and (2) is a physical argument. A force acting on the mass over time changes its momentum. In fact, the best way to state Newton's second law is that

$$\frac{dp}{dt} = f(t),$$

where $p(t)$ is the momentum of a system and $f(t)$ is an external force acting on the system. If a force $f(t)$ acts over the time interval $[t_1, t_2]$ the total change of momentum due to the force is

$$\int_{t_1}^{t_2} f(t)\, dt.$$

Physicists call this the impulse of the force $f(t)$ over the interval $[t_1, t_2]$. If a very large force is applied over a very short time interval and has total impulse of 1 the result will be a sudden unit jump in the momentum of the system.

For a second order system the unit impulse function $\delta$ can be thought of as an idealization of this force. It is a force with total impulse 1 applied all at once.

A third argument that we will skip would be to solve equation (1) with a box function for input and take the limit as the box gets narrower and taller always with area 1.

## 2. Unit Impulse Response

We consider once again the damped harmonic oscillator equation

$$m\ddot{x} + b\dot{x} + kx = f(t).$$

The **unit impulse response** is the solution to this equation with input $f(t) = \delta(t)$ and rest initial conditions: $x(t) = 0$ for $t < 0$. That is, it is the solution to the initial value problem (IVP)

$$m\ddot{x} + b\dot{x} + kx = \delta(t), \quad x(0^-) = 0, \, \dot{x}(0^-) = 0.$$

This could be a damped spring-mass system with mass $m$, damping constant $b$ and spring constant $k$. The mass is at rest at equilibrium until time $t = 0$ when it is hit by a sudden very brief very intense force, rather like getting hit on the head by a hammer. The effect is to increase the momentum instantaneously, without changing the position of the mass.

Let $w(t)$ denote the solution we seek. The rest initial conditions tell us that $w(t) = 0$ for $t < 0$. We know from section 1 that the effect of the input is to cause a unit jump in the momentum at $t = 0$ and no change in position. We also know that, for $t > 0$, the input $\delta(t) = 0$. Putting this together, for $t > 0$ the $w(t)$ satisfies the equation

$$m\ddot{w} + b\dot{w} + kw = 0, \quad \dot{w}(0^+) = 1/m, \, w(0^+) = 0.$$

This is a homogeneous constant coefficient linear differential equation which we have lots of practice in solving.

**Example 1.** Find the unit impulse response for the system

$$2\ddot{x} + 8\dot{x} + 26x = f(t). \tag{2}$$

**Solution.** We will use the standard notation $w(t)$ for the unit impulse response. We are looking for the response from rest to $f(t) = \delta(t)$. We know

$w(t) = 0$ for $t < 0$. At $t = 0$ the input causes a unit jump in momentum, i.e., $2\dot{w}(0^+) = 1$. So, for $t > 0$ we have to solve

$$2\ddot{w} + 8\dot{w} + 26w = 0, \quad \dot{w}(0^+) = 1/2,\ w(0^+) = 0.$$

The roots of the characteristic polynomial are $-2 \pm 3i$. Which implies

$$w(t) = c_1 e^{-2t}\cos(3t) + c_2 e^{-2t}\sin(3t), \qquad \text{for } t > 0.$$

The initial conditions give

$$\begin{aligned} 0 &= w(0^+) = c_1, \\ 1/2 &= \dot{w}(0^+) = -2c_1 + 3c_2 \ \Rightarrow\ c_2 = 1/6. \end{aligned}$$

Thus, the unit impulse response (in both cases and $u$-format) is

$$w(t) = \begin{cases} 0 & \text{for } t < 0 \\ \frac{1}{6}e^{-2t}\sin(3t) & \text{for } t > 0 \end{cases} \quad = \quad \frac{1}{6}e^{-2t}\sin(3t)u(t). \tag{3}$$

Figure 1 the graph of the unit impulse response. Notice that at $t = 0$ the graph has a corner. This corresponds to the slope $\dot{w}$ jumping from 0 to 1/2. For $t > 0$ the graph decays to 0 while oscillating.

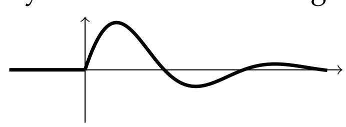

Figure 1. The unit impulse response for the system $2\ddot{x} + 8\dot{x} + 26x$.

## 3. Checking Example 1 by Substitution

With any differential equation you can verify a solution by plugging it into the equation. We will do that with example 1 to gain some more insight into why we get the solution.

First, we take the derivatives of the solution in equation (3) for $t \neq 0$

$$\dot{w}(t) = \begin{cases} \frac{1}{6}e^{-2t}(-2\sin(3t) + 3\cos(3t)) & \text{for } t > 0 \\ 0 & \text{for } t < 0 \end{cases} \quad = \ \tfrac{1}{6}e^{-2t}\left(-2\sin(3t) + 3\cos(3t)\right)\, u(t)$$

$$\ddot{w}(t) = \begin{cases} \frac{1}{6}e^{-2t}(-5\sin(3t) - 12\cos(3t)) & \text{for } t > 0 \\ 0 & \text{for } t < 0 \end{cases} \quad = \ \tfrac{1}{6}e^{-2t}\left(-5\sin(3t) - 12\cos(3t)\right)\, u(t)$$

Next we look at the jumps at $t = 0$

$$\begin{aligned} w(0^-) &= 0, & w(0^+) &= 0 \\ \dot{w}(0^-) &= 0, & \dot{w}(0^+) &= 1/2 \end{aligned}$$



$t$

Now we can compute the full generalized derivatives (we just give them in $u$-format)

$$\begin{aligned} \dot{w}(t) &= \tfrac{1}{6}e^{-2t}\left(-2\sin(3t)+3\cos(3t)\right)\, u(t) \\ \ddot{w}(t) &= \tfrac{1}{2}\delta(t) + \tfrac{1}{6}e^{-2t}\left(-5\sin(3t)-12\cos(3t)\right)\, u(t) \end{aligned}$$

Finally we substitute $w$ for $x$ in equation (2)

$$\begin{array}{lcl} 2\ddot{w}(t) & = & \delta(t) - \tfrac{-5}{3}e^{-2t}\sin(3t) - 4e^{-2t}\cos(3t) \\ 8\dot{w}(t) & = & \tfrac{-8}{3}e^{-2t}\sin(3t) + 4e^{-2t}\cos(3t) \\ 26w(t) & = & \tfrac{13}{3}e^{-2t}\sin(3t) \\ \hline 2\ddot{w}+8\dot{w}+26w & = & \delta(t). \end{array}$$

**The Meaning of the Phrase 'Unit Impulse Response'**

As we've noted several times already, the response to a given input depends on what we consider to be the input. For example, if our system is

$$m\ddot{x} + b\dot{x} + kx = b\dot{y}$$

and we consider $y$ to be the input, then the unit impulse response is the solution to

$$m\ddot{x} + b\dot{x} + kx = b\dot{\delta}(t) \quad \text{with rest IC.}$$

(Here, $\dot{\delta}$ is what we've called a *doublet*.) For $t > 0$ this is equivalent to

$$m\ddot{x} + b\dot{x} + kx = 0 \quad \text{with post IC} \quad x(0^+) = \frac{b}{m}, \quad \dot{x}(0^+) = -\frac{b^2}{m^2}$$