18.03SC Practice Problems 25

Step and delta responses

Solution suggestions

1. *Find the unit step and unit impulse responses for the operator $2D + I$, and graph them.*

The unit step response $h = h(t)$ is the continuous solution that is zero for $t < 0$, and is a solution of

$$2\dot{x} + x = 1 \quad \text{for } t > 0. \tag{1}$$

This equation has particular solution $x_p = 1$. The homogeneous system $2\dot{x} + x = 0$ has general solution $ce^{-t/2}$, so the general solution of (1) is $x = 1 + ce^{-t/2}$.

Because there is no impulse at $t = 0$ the pre and post-initial conditions are the same, i.e. $x(0^-) = x(0^+) = 0$. We need to choose the constant c to fit the post-initial condition: $x(0^+) = 1 + c = 0$. Thus, $c = -1$ and the unit step response h is

$$h(t) = \left(1 - e^{-t/2}\right) u(t).$$

The unit impulse response $w = w(t)$ is the solution that is zero for $x < 0$, a solution of $2\dot{x} + x = 0$ for $x > 0$, and satisfies $x(0^+) = 1/a_1 = 1/2$, where a_1 is the coefficient of $\dot{x}$. Or, alternatively, but equivalently, the unit impulse response is the derivative of the unit step response, so, using the product rule

$$w(t) \;=\; h'(t) \;=\; \frac{1}{2}e^{-t/2}\,u(t) + \left(1 - e^{-t/2}\right)\delta(t) \;=\; \frac{1}{2}e^{-t/2}\,u(t).$$

The term $(1 - e^{-t/2})\delta(t) = 0$ because at $t = 0$ the coefficient of $\delta(t)$ is 0. The graphs of both are given below.

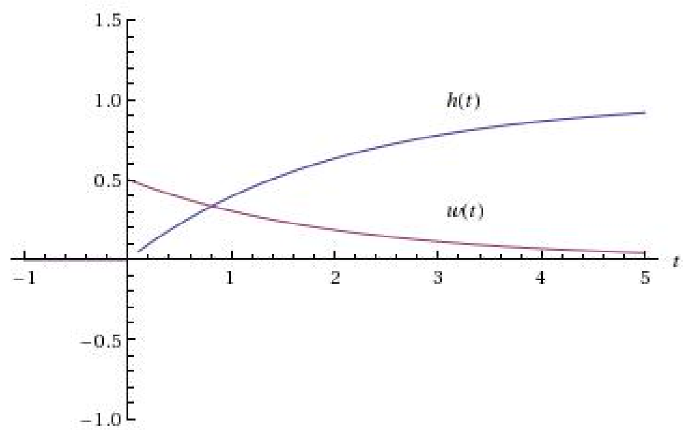

Figure 1: The unit step response $h(t)$ and unit impulse response $w(t)$ for $2D + I$.

2. *Find the unit impulse response for the operator $D^2 + 2D$, and graph it.*

The unit impulse response for this operator is the function $w(t)$ that is zero for $t < 0$ and satisfies the equation

$$\ddot{x} + 2\dot{x} = 0$$

for $t > 0$ with post initial conditions $x(0^+) = 0$ and $\dot{x}(0^+) = 1/1 = 1$, since the operator is of order 2 and has leading coefficient 1.

By examining the characteristic polynomial, we see that homogeneous solutions have the form $c_1 e^{-2t} + c_2$.

Now we use the post initial conditions to find the right constants c_1 and c_2. From the condition on the function itself, $c_1 + c_2 = 0$, and, from the condition on the first derivative, $-2c_1 = 1$. Thus, $c_1 = -1/2$, $c_2 = -c_1 = 1/2$, and the unit impulse response is

$$w(t) = \frac{1}{2}\left(1 - e^{-2t}\right) u(t).$$

The graph of $w(t)$ is given below.

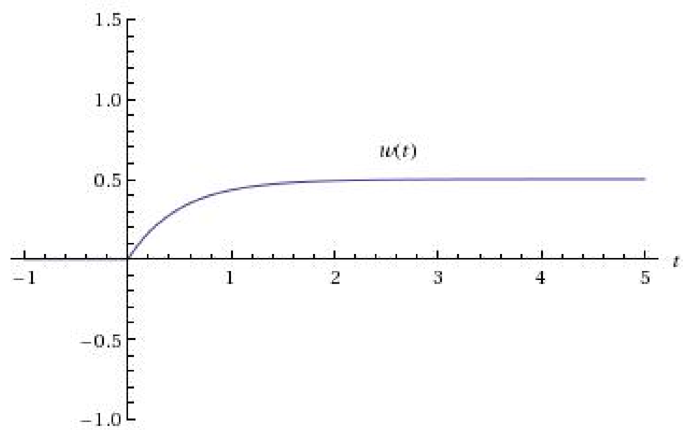

Figure 2: The unit impulse response $w(t)$ for $D^2 + 2D$.

3. *From your answer to* **2.**, *find the solution to* $\ddot{x} + 2\dot{x} = 3\delta(t-1)$ *with rest initial conditions.*

Using time invariance, we find that a solution to $\ddot{x} + 2\dot{x} = 3\delta(t-1)$ is

$$x = 3w(t-1) = \frac{3}{2}\left(1 - e^{2-2t}\right) u(t-1).$$