Part II Problems and Solutions

Problem 1: *[Step and delta responses]*

(a) *Find the unit impulse response* w *for the LTI operator* $2D^2 + 4D + 4I$.

(b) *Find the unit step response* v *for the same operator.*

(c) *Verify that* $\dot{v} = w$ *(as it should be, since* $\dot{u} = \delta$*).*

(d) *For each of the following functions, find the LTI differential operator* $p(D)$ *having it as unit impulse response.*

(i) $2u(t)$.

(ii) $u(t)t$.

(iii) $u(t)t^2$.

Solution: **(a)** The roots of the characteristic polynomial are $-1 \pm i$, so the general solution to the homogeneous equation is $e^{-t}(a \cos t + b \sin t)$. The unit impulse response for this second order operator has $w(0) = 0$ and $\dot{w}(0+) = \frac{1}{2}$. The first forces $a = 0$ and the second gives $b = \frac{1}{2}$: $w(t) = \frac{1}{2}u(t)e^{-t} \sin t$.

(b) For $t > 0$, the unit step response is a solution to $p(D)x = 1$. In our case, $x_p = \frac{1}{4}$ is such a solution, and the general solution is then $x = \frac{1}{4} + e^{-t}(a \cos t + b \sin t)$. We require rest initial conditions: $0 = x(0) = \frac{1}{4} + a$ or $a = -\frac{1}{4}$. $\dot{x} = e^{-t}((-a + b) \cos t + (-a - b) \sin t)$, so $0 = \dot{x}(0) = -a + b$ and $b = -\frac{1}{4}$ as well: $v = \frac{1}{4}u(t)(1 - e^{-t}(\cos t + \sin t))$.

(c) $\dot{v} = -\frac{1}{4}e^{-t}((-1 + 1) \cos t + (-1 - 1) \sin t) = \frac{1}{2}e^{-t} \sin t$.

(d) (i) This function has a jump in *value*, so the operator must be of first order. $(aD + bI)(2u) = 2a\delta(t) + 2bu(t)$, so $b = 0$ and $a = \frac{1}{2}$: $p(D) = \frac{1}{2}D$.

(ii) This function has no jump but its derivative does, so the operator must be of second order. For $t > 0$, $w(t) = t$ is the solution to $a_2\ddot{x} + a_1\dot{x} + a_0 x = 0$ with $x(0) = 0$ and $\dot{x}(0) = \frac{1}{a_2}$. Plug in: $a_1 + a_0 t = 0$ implies $a_1 = a_0 = 0$, and $1 = \frac{d}{dt}t\Big|_{t=0} = \frac{1}{a_2}$ implies that $a_2 = 1$. So $p(D) = D^2$. Or you can argue that $w(t) = u(t)t$, $\dot{w}(t) = u(t)$ and $\ddot{w}(t) = \delta(t)$, so $a_2\delta(t) = \delta(t)$ and $a_2 = 1$.

(iii) This function $w(t)$ has no jump in value or derivative, but its second derivative does jump: $\ddot{w}(t) = 2u(t)$. So $w^{(3)}(t) = 2\delta(t)$. This means that we are looking for a third order operator, $a_3D^3 + a_2D^2 + a_1D + a_0I$. t^2 is a solution to the homogeneous equation, so $a_2 \cdot 2 + a_1 \cdot 2t + a_0t^2 = 0$, which implies that $a_0 = a_1 = a_2 = 0$. $\ddot{w}(0) = 2$ implies that $a_3 = \frac{1}{2}$ and $p(D) = \frac{1}{2}D^3$. Or you can argue that $w(t) = u(t)t^2$, $\dot{w}(t) = u(t)2t$, $\ddot{w}(t) = 2u(t)$, $w^{(3)}(t) = 2\delta(t)$, so $a_3w^{(3)}(t) = \delta(t)$ implies that $a_3 = \frac{1}{2}$.