18.03SC Differential Equations, Fall 2011
Transcript – Pole Diagrams

PROFESSOR: Welcome. In this session, we're going to examine pole diagrams. So let's consider a pole diagram for the linear time independent systems of the form bdy equals f. So here, we have a few pole diagrams to examine with the poles marked with the red crosses. The axes are imaginary axis and the real axis.

And the questions are list all the stable systems. Choose the systems with the fastest decay. And last, choose fastest decay without oscillation. So you have to think of how to interpret these diagrams and what are the meanings of the position of the different poles to address these questions. So pause the video. Take a few minutes. And I'll be right back.

Welcome back. So the first question asks us to list all the stable systems. So if you recall, all the stable systems would be the ones for which the real part of the pole are all negative. So if we have all the poles with real part negative, we would have a stable system for which the solution after a long time would decay, basically. It would be transient.

So here, if we look at our pole diagrams, we have diagram B for which the real part of the two poles are minus 3 and minus 2. So this would definitely generate a decaying exponential, exponential minus 2t, exponential minus 3t type function at large time for this system. In A, we have a minus 1 and a 1. So the real part here would give us a decay, an exponential decay, exponential minus t. But we would have here a term that will diverge with exponential t. So this would have basically behavior a long time that diverges and doesn't go to 0. So this would not basically be a stable system.

For C, we have all the real parts negative. So we're in the stable case. For D, we also have the same. All the real parts are negative. So these are imaginary, 3i minus 2 plus 3i. So both of them have also negative real parts. Everything is on the left side of the plane. Same thing for E. And for F, we're right on the imaginary axis, which means that we can't tell if it's going to be an exponential. It's not going to be an exponential decay or growth at this point.

So to answer the first question, the stable systems would be B, C, D, E, and that's it. So now, for the second part, choose the system with the fastest decay. So here, again, as we just saw, the real part of the poles gives us the speed at which the system is going to decay. And so we have to look for a system that has its pole on the rightmost part of the plane.

So for example, if we restrict ourselves here, the fastest decay here is going to be governed by the rightmost pole, so exponential minus 2t that we need to examine. Here, we have an exponential minus 3t decay time. Here, we would have an exponential minus t decay. Here, we would have something a little bit-- minus 0.5t. And here, again, this is not stable system. We did decay at large time.

So which of the system has the fastest decay? We would need to select the one with the rightmost pole that is the most on the left. So we would need to compare this minus 2, with this minus 3, with this minus 1, with this minus 0.5. And from all these poles, the one that has the minus 3 here would be the one that would be decaying the fastest.

So we would have system C decaying as exponential minus 3t. And I'm going to just write down the others. And these would be the dominant terms that would basically subscribe the behavior at large t. So this is the fastest. For our system B, which is not the fastest, I'm just going to write it, it's going to be dominated by a minus 2t. For the system D, it would be dominated by a minus 1t. For the system E, it would be dominated by the minus 0.5t. So the fastest decay would definitely be system C.

So the last question, choose the fastest decay without oscillations. So now, we need to pay attention to whether the poles are on the imaginary axis, on the real axis, or somewhere in the middle. So let's again, examine only the stable cases. So here, the two poles are in the real axis. So it's going to give us a behavior that is only pure decay, exponential decay, exponential minus 2t, exponential minus 3t with no oscillation. Because basically, these two people don't have any imaginary parts.

Here, we have a term in exponential minus 3t. But we would have a term that would show oscillations with the circular frequencies of 3. So you would have an exponential minus 3t cosine 3t plus exponential minus 3t sine 3t, for example. So here, we would have a system that shows oscillation. So we don't want that.

For this case, that would be similar. So you see the pattern. Here, we have an exponential decay. Here, we would have decaying oscillations. So there is oscillations. Same thing here. And so we're left with only system B, which basically does not show any oscillation but only an exponential decay. Pure decay. And it's dominated by the exponential minus 2t without oscillations. So that ends this first part of the problem. I'm just going to take a minute and come back to show you the second part of the problem.

Welcome back for the second part of the problem. So here, another pole diagram that we're going to label J for this system. And the poles are basically here, 0.5 plus 4i, 0.5 minus 4i. And then here, we have almost a minus 3. And here, it would be a minus 4. So the question is to consider now the previous system we had, p of dy equals f of t. But now, with f of t equals f0 cos omega t, and omega, the circular frequency going between one and five.

And the question is for which values of omega do we have the largest response from the system G? So basically, we're looking at phenomenon of resonance where we want to find the frequency omega for which the response of the system is going to be the biggest. So I give you just a few minutes, and we'll be right back.

Welcome back for this second part. So here, if you recall, the amplitude response from the previous recitations, the amplitude is governed by 1 over the characteristic polynomial evaluated at i omega. Where does i omega come from? If you recall, it comes from the exponential response formula. Because on the right-hand side here, we have a cosine omega t, then we complexify exponential and we get, et cetera, to get the amplitude response.

So the amplitude is going to be the largest for basically values of i omega that are the closest to the poles of the polynomial to the poles of this function. Basically, the poles of the 1 over characteristic polynomial is going to give us the 1 over 0, which will give an amplitude response that goes to infinity, so the largest response. So we want to choose our frequency, i omega. Basically, we want to look at this diagram and look at the poles that are the closest to the values that are going to give us p of s equals to 0.

And this is basically diagrammed that gives us the pole of this 1 over p. And so we want to choose a value for i omega, which will be on the imaginary axis, that is the closest to one of these two poles. Given that we work with positive frequencies, we would want, for example, to choose i omega equals i4. So it would be here, or just omega equals to 4.

And this value of omega would get us very close to this pole, which would mean then that we are 1 over a very small number, which means that the amplitude response will be the biggest. So that's the look behind the resonance and looking at, again, the phenomenon of the resonance from the pole diagram perspective and with the idea of the use of the laplace transform behind the whole technique.

So that ends this session. And the goal was really to learn to interpret the meaning of the poles in the pole diagram and how do the polls give us an idea inside on the behavior of the system on the long term.