**18.03SC Differential Equations, Fall 2011**
**Transcript – Partial Fractions and Laplace Inverse**

PROFESSOR: Welcome back. So today we're going to take a look at a problem with partial fractions, and specifically how to use these partial fractions to compute Laplace inverses. So just as a warm-up, we're asked to recall what the formula is for the Laplace transform of f prime, in terms of the Laplace transform of f.

In the second part, we're asked to find the inverse Laplace transform for three different functions. The first one is 1 over s squared minus 4, s squared divided by s squared plus 4, e to the minus 5s divided by s square minus 4. And notice how none of these functions appear in the look-up table for Laplace transform. So in each case, we have to use partial fractions to convert it or massage these functions into something which we do know the Laplace transform inverse for.

And then lastly, for the third problem we're asked to write down the partial fraction decomposition of this function, 1 over s squared times s squared plus 4 times s plus 1 times s plus 3. And specifically, when we do the partial fraction decomposition, we're just going to leave the undetermined coefficients unknown. We're not actually going to solve for them.

OK? So I'll let you think about this problem. And I'll be back in a moment. Hi everyone. Welcome back.

OK. So for part one, we're just asked to recall the Laplace transform of f prime in terms of the Laplace transform of f. So the Laplace transform of f prime we can write down as s times the Laplace transform of F, which I'll use capital F, minus f evaluated at the lower bound of the Laplace transform integral, which in this case is just zero minus. OK. So this is just part one. This is a warm-up problem.

For part two, we're asked to find the inverse Laplace transform for three different functions. The first one is 1 over s squared minus 4. And we see here that we can factor the denominator into one over s minus 2 times 1 over s plus 2, which means that we can use a partial fraction decomposition which has the form of A divided by s minus 2 plus B divided by s plus 2. And then we need to solve for the coefficients A and B.

So one way to solve for the coefficients A and B is just to multiply both sides of this equation through by the factors s minus 2 and s plus 2. We can then plug in values of s and solve for A and B. There's another way, which is sometimes referred to a cover-up method. And in this case, what we do is we pick, for example, say one of the places where the denominator blows out. So for example, if we look at this factor s minus 2, this factor diverges when s approaches 2. So then what we do is we go back to our original function, we cover up the term s minus 2 where it diverges, and then in the remaining term, we plug in the value of s, which causes the factor s minus 2 to diverge.

So in this case, s minus 2 diverges at 2. So we would cover this factor up and then plug in s is equal to 2. And this would give us the value of A. So for this problem, A is equal to 1 over 2 plus 2, which is just 1/4.

For B, we look at plugging in s equals negative 2. So we cover up the factor that diverges in the original function. Plugging in s equals negative 2 gives me negative 2 minus negative 2. So B is just negative 4.

So this function is 1/4 1 over s minus 2 minus 1/4 1 over s plus 2. And when we take the inverse Laplace transform, so the inverse Laplace transform of 1 over s squared minus 4 is going to be 1/4. The inverse Laplace transform of s minus 2 is e to the plus 2t. The second factor is negative 1/4 e to the negative 2t. And this concludes the first problem.

For the second function, we have s squared divided by s squared plus 4. So again, this function's not in the correct form to use a Laplace transform look-up table. But what we can do is we can divide long division of polynomials the numerator out by the denominator.

So when we do that we end up with 1 minus 4 divided by s squared plus 4. So when we take the inverse Laplace transform of s squared divided by s squared plus 4, we're left with the inverse Laplace transform of 1 minus the inverse Laplace transform of 4 divided by s squared plus 4. And if we use our look-up table, we know that the inverse Laplace transform of 1 is the delta function.

Meanwhile, the inverse Laplace transform of some number divided by s squared plus 4 looks like, in this case, sine of 2t. Now we need a 2 upstairs, so this is going to give us 2 times sine of 2t. And I obtain this by noting that the Laplace transform of sine omega t is equal to omega divided by s squared plus omega squared. So that's how I went from this function, this Laplace transform inverse, to 2 sine 2t.

Lastly, for part three, the third problem, we're asked for the inverse Laplace transform of e to the minus 5s divided by s squared minus 4. Now note that we already know the inverse Laplace transform to 1 over s squared minus 4. In this case, we're just multiplying 1 over s squared minus 4 by e to the minus 5s. So we can use the shift formula in addition to what we already computed, the inverse Laplace transform to 1 over s squared minus 4, to compute this inverse Laplace transform.

So the shift formula says that we need to multiply by the step function t minus 5. So we're shifting with the step function. And then what we do is we want to shift the inverse Laplace transform of 1 over s squared minus 4. So the inverse Laplace transform to 1 over s squared minus 4 is 2 t minus 5.

So if we shift t by 5, we end up with 1/4 e to the 2 t minus 5 minus 1/4 e to the minus 2 t minus 5. So again, just to reiterate, when we have an exponential e to the minus, say, 5s and we already know the inverse Laplace transform of everything else, we just multiply by the step function u t minus 5, the same shift. And then wherever we see the t in the inverse Laplace transform, we just replace it with t minus 5, in this case. OK?

And now lastly, for part three we're just asked to write down the decomposition for this very large function, the partial fraction decomposition of this large function. So it's s squared times s squared plus 1 times s plus 1 times s plus 3. Let me just double check. s squared plus 4. OK?

And unfortunately this function's going to be a little ugly. We're going to have a constant, A, divided by s plus B divided by s squared. So these first two terms come from this term right here.

For the squared plus 4 term, we're going to seek a C times s plus D-- so we need both factors, Cs plus D-- divided by s squared plus 4. We have E divided by s plus 1. And then plus F divided by s plus 3. So just to associate each factor in the partial fraction decomposition with the original function, I've just drawn some arrows. OK?

And then lastly although I didn't state this in the original problem, we can also just write down the inverse Laplace transform of this entire mess without actually determining the constants. We can just keep the constants in. So if we wanted to compute the inverse Laplace transform, well, it would just be A-- because inverse Laplace transform of 1 over s is 1-- plus B-- inverse Laplace transform of s squared is t-- plus, now this part has two terms, Cs divided by s squared plus 4. That's going to give us C times cosine of 2t.

The D divided by s squared plus 4 is going to give us, well, we need to multiply upstairs by 2 and divide downstairs by 2. So we get D divided by 2 of sine 2t. This term, E, is going to give us e times the exponential of negative t. And then the last term is going to give us F times e to the minus 3t. OK?

So just as a quick wrap-up, in this problem we've computed partial fraction decompositions of several functions. And the reason we've done this is because we often end up in a situation when we want to solve an ODE using Laplace transforms where we have some complicated function and we need to use partial fraction decomposition to write down simpler functions, in which case we can then use the Laplace inverse formula on each of the simpler functions. OK? So I'll just conclude here, and I'll see you next time.