

Table Entries: Repeated Quadratic Factors

1. Repeated Quadratic Factors

We will add three entries to our Laplace Table.

$$\mathcal{L}\left(\frac{1}{2\omega^3}(\sin(\omega t) - \omega t \cos(\omega t))\right) = \frac{1}{(s^2 + \omega^2)^2} \quad (1)$$

$$\mathcal{L}\left(\frac{t}{2\omega} \sin(\omega t)\right) = \frac{s}{(s^2 + \omega^2)^2} \quad (2)$$

$$\mathcal{L}\left(\frac{1}{2\omega}(\sin(\omega t) + \omega t \cos(\omega t))\right) = \frac{s^2}{(s^2 + \omega^2)^2} \quad (3)$$

There are several ways to prove these formulas. We will give one using partial fractions by factoring the denominators on the frequency side into complex linear factors.

Proof of 1. First some algebra:

$$\frac{1}{(s-a)^2(s+a)^2} = \frac{A}{(s-a)^2} + \frac{B}{s-a} + \frac{C}{(s+a)^2} + \frac{D}{s+a}$$

Cover-up gives us A and C . Undetermined coefficients then gives B and D :

$$A = \frac{1}{4a^2} = C, \quad D = \frac{1}{4a^3} = -B$$

This gives the inverse Laplace transform

$$\mathcal{L}^{-1}\left(\frac{1}{(s-a)^2(s+a)^2}\right) = \frac{1}{4a^2}(te^{at} + te^{-at}) - \frac{1}{4a^3}(e^{at} - e^{-at}). \quad (4)$$

We will use this on the right hand side of (1), but first recall

$$e^{i\omega t} + e^{-i\omega t} = 2 \cos(\omega t) \quad \text{and} \quad e^{i\omega t} - e^{-i\omega t} = 2i \sin(\omega t) \quad (5)$$

Let $a = i\omega$, then (4) and (5) combine to prove formula (1).

$$\begin{aligned} \mathcal{L}^{-1}\left(\frac{1}{(s^2 + \omega^2)^2}\right) &= \mathcal{L}^{-1}\left(\frac{1}{(s-i\omega)^2(s+i\omega)^2}\right) \\ &= -\frac{1}{4\omega^2}(te^{i\omega t} + te^{-i\omega t}) + \frac{1}{4\omega^3 i}(e^{i\omega t} - e^{-i\omega t}). \\ &= -\frac{1}{2\omega^2}t \cos(\omega t) + \frac{1}{2\omega^3} \sin(\omega t). \end{aligned}$$

The proofs of (2) and (3) are similar, and we will omit them.

2. Note on the Relation to Resonance:

Each of the formulas (1), (2), and (3) has a term with a factor of t . This is exactly what we saw with the response x in the resonance equation

$$\ddot{x} + \omega^2 x = \cos(\omega t),$$

which has solution $x(t) = t \sin(\omega t)/(2\omega)$.

Notice that $\mathcal{L}^{-1}(1/s^2) = t$ and the s -shift rule shows $\mathcal{L}^{-1}(1/(s-a)^2) = te^{at}$. So repeated factors on the frequency side always lead to multiplication by t on the time side. If the repeated factor has a higher power then we get multiplication by a higher power of t .

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