Application to Infinite Series

There is a famous formula found by Euler:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}. \tag{1}$$

We'll show how you can use a Fourier series to get this result.

Consider the period 2π function given by $f(t) = t\left(\pi - \frac{t}{2}\right)$ on $[0, 2\pi]$.

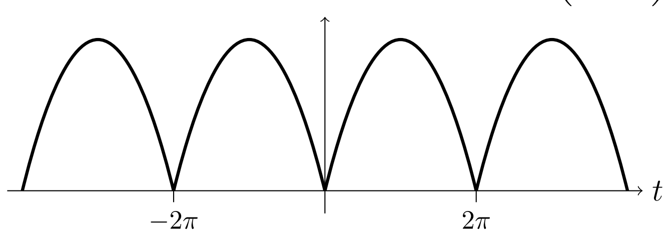

Figure 1: Graph of $f(t)$.

First, we compute the Fourier series of $f(t)$. Since f is even, the sine terms are all 0. For the cosine terms it is slightly easier to integrate over a full period from 0 to 2π rather than doubling the integral over the half-period. We give the results, but leave the details of the integration by parts to the reader.

For $n = 0$ we have

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} t(\pi - t/2)\,dt = \frac{2\pi^2}{3}$$

and for $n \neq 0$ we have

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{2\pi} t(\pi - t/2)\cos(nt)\,dt \\ &= \frac{1}{\pi}\left[\frac{\pi t\sin(nt)}{n} + \frac{\pi\cos(nt)}{n^2} - \frac{t^2\sin(nt)}{2n} - \frac{t\cos(nt)}{n^2} + \frac{\sin(nt)}{n^3}\right|_0^{2\pi} = -\frac{2}{n^2}. \end{aligned}$$

Thus the Fourier series is $f(t) = \frac{\pi^2}{3} - 2\sum_{n=1}^{\infty}\frac{\cos(nt)}{n^2}$.

Since the function $f(t)$ is continuous, the series converges to $f(t)$ for all t. Plugging in $t = 0$, we then get

$$f(0) = 0 = \frac{\pi^2}{3} - \sum_{n=1}^{\infty}\frac{2}{n^2}.$$

A little bit of algebra then gives Euler's result (1).

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