Convergence of Fourier Series

The period $2L$ function $f(t)$ is called **piecewise smooth** if there are a only finite number of points $0 \leq t_1 < t_2 < \ldots < t_n \leq 2L$ where $f(t)$ is not differentiable, and if at each of these points the left and right-hand limits $\lim_{t \to t_i^+} f'(t)$ and $\lim_{t \to t_i^-} f'(t)$ exist (although they might not be equal).

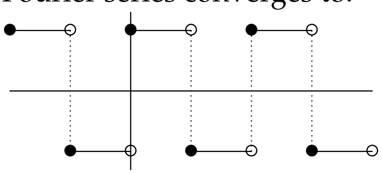
Recall that when we first introduced Fourier series we wrote

$$\begin{aligned} f(t) &\sim \frac{a_0}{2} + a_1 \cos(t) + a_2 \cos(2t) + a_3 \cos(3t) + \ldots \\ &\quad + b_1 \sin(t) + b_2 \sin(2t) + b_3 \sin(3t) + \ldots \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt) + b_n \sin(nt), \end{aligned}$$

where we used '$\sim$' instead of an equal sign. The following theorem shows that our subsequent use of an equal sign, while not technically correct, is close enough to be warranted.

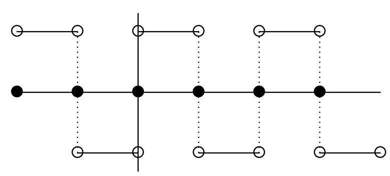
Theorem: If $f(t)$ is piecewise smooth and periodic then the Fourier series for f
1. converges to $f(t)$ at values of t where f is continuous
2. converges to the average of $f(t^-)$ and $f(t^+)$ where it has a jump discontinuity.

Example. Square wave. No matter what the endpoint behavior of $f(t)$ the Fourier series converges to:



Original $f(t)$

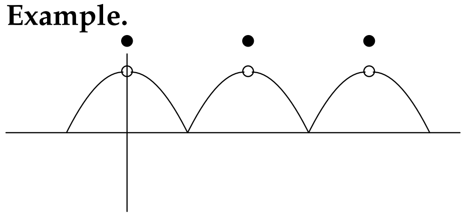
$\rightsquigarrow$



Fourier series for $f(t)$

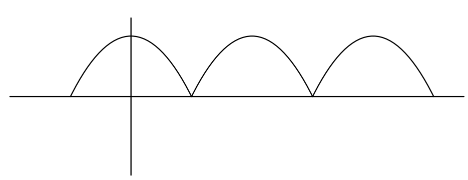
Example. Continuous sawtooth: Fourier series converges to $f(t)$.

Example.



Original $f(t)$

$\rightsquigarrow$



Fourier series