Even and Odd Functions

If a periodic function $f(t)$ is an even function we have already used the fact that its Fourier series will involve only cosines. Likewise the Fourier series of an odd function will contain only sines. Here we will give short proofs of these statements.

Even and odd functions.
Definition. A function $f(t)$ is called **even** if $f(-t) = f(t)$ for all t.

The graph of an even function is symmetric about the y-axis. Here are some examples of even functions:

1. $t^2, t^4, t^6, \ldots$, any *even* power of t.

2. $\cos(at)$ (recall the power series for $\cos(at)$ has only even powers of t).

3. A constant function is even.

We will need the following fact about the integral of an even function over a 'balanced' interval $[-L, L]$.

$$\text{If } f(t) \text{ is even then} \qquad \int_{-L}^{L} f(t)\,dt = 2\int_0^L f(t)\,dt.$$

This fact becomes clear if we think of the integral as an area (see fig. 1).

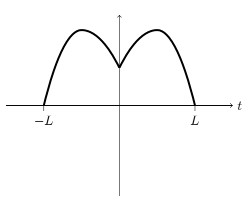

Fig. 1: Even functions:
(total area = twice area of right half)

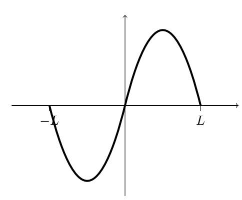

Fig. 2: Odd functions:
(total (signed) area is 0)

Definition. A function $f(t)$ is called **odd** if $f(-t) = -f(t)$ for all t.

The graph of an odd function is symmetric about the the origin. Here are some examples of odd functions:

1. $t, t^3, t^5, \ldots$, any *odd* power of t.

2. $\sin(at)$ (recall the power series for $\sin(at)$ has only odd powers of t).

We will need the following fact about the integral of an odd function over a 'balanced' interval $[-L, L]$.

$$\text{If } f(t) \text{ is odd then} \quad \int_{-L}^{L} f(t)\,dt = 0.$$

This fact becomes clear if we think of the integral as an area (see Fig. 2).

Multiplying Even and Odd Functions

When multiplying even and odd functions it is helpful to think in terms of multiply even and odd powers of t. This gives the following rules.

1. even × even = even
2. odd × odd = even
3. odd × even = odd

All this leads to the **even and odd Fourier coefficient rules**:

Assume $f(t)$ is periodic then:

1. If $f(t)$ is even then we have $b_n = 0$, and $a_n = \dfrac{2}{L}\displaystyle\int_0^L f(t)\cos\left(n\frac{\pi}{L}t\right)\,dt$.
2. If $f(t)$ is odd then we have $a_n = 0$, and $b_n = \dfrac{2}{L}\displaystyle\int_0^L f(t)\sin\left(n\frac{\pi}{L}t\right)\,dt$.

Reason: Assume $f(t)$ is even. The rule for multiplying even functions tells us that $f(t)\cos at$ is even and the rule for integrating an even function over a symmetric interval tell us that

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(n\frac{\pi}{L}t\right)\,dt = \frac{2}{L}\int_0^L f(t)\cos\left(n\frac{\pi}{L}t\right)\,dt.$$

Likewise, the rule even × odd = odd tell us that $f(t)\sin at$ is odd, and so the integral for b_n is 0.

If $f(t)$ is odd everything works much the same. The rule for multiplying odd functions tells us that $f(t)\sin at$ is even and therefore

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(n\frac{\pi}{L}t\right)\,dt = \frac{2}{L}\int_0^L f(t)\sin\left(n\frac{\pi}{L}t\right)\,dt.$$

Likewise the rule odd × even = odd tells us that $f(t)\cos(at)$ is odd, and so the integral for a_n is 0.

Examples: In previous sessions we saw the odd square wave had only sine coefficients and the even triangle wave had only cosine coefficients.