Compute a Fourier Series

Exercise. We warm up with a reminder of how one computes the Fourier series of a given periodic function using the integral Fourier coefficient formulas.

Compute the Fourier series for the period 2π continuous sawtooth function $f(t) = |t|$ for $-\pi \le t \le \pi$.

Answer.

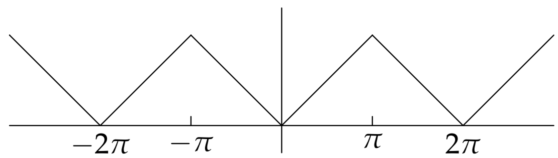

Figure 1. Graph of the period 2π continuous sawtooth function.

The period is 2π, so the half-period $L = \pi$. Since $f(t) = |t|$ for $-\pi \le t \le \pi$, it is an even function we know the Fourier sine coefficents b_n must be zero.

Computing the cosine coefficients we get: For $n \ne 0$:

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} |t|\cos(nt)\,dt = \frac{2}{\pi}\int_0^{\pi} t\cos(nt)\,dt \\ &= \frac{2}{\pi}\left(\frac{t\sin(nt)}{n} + \frac{\cos(nt)}{n^2}\right|_0^{\pi} = \frac{2}{n^2\pi}((-1)^n - 1) = \begin{cases} -\frac{4}{n^2\pi} & \text{for } n \text{ odd} \\ 0 & \text{for even} \end{cases} \end{aligned}$$

For $n = 0$:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |t|\,dt = \frac{2}{\pi}\int_0^{\pi} t\,dt = \pi.$$

Thus, $f(t)$ has Fourier series

$$\begin{aligned} f(t) &= \frac{\pi}{2} - \frac{4}{\pi}\left(\cos t + \frac{\cos(3t)}{3^2} + \frac{\cos(5t)}{5^2} + \cdots\right) \\ &= \frac{\pi}{2} - \frac{4}{\pi}\sum_{n \text{ odd}} \frac{\cos(nt)}{n^2} \end{aligned}$$

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