Part II Problems and Solutions

Problem 1: *[Fourier Series]*

(a) *Find the Fourier series for* $2\sin(t - \frac{\pi}{3})$ *(Hint: A function of period* 2π *has just one expression as a linear combination of* $\cos(mt)$*'s and* $\sin(nt)$*'s.)*

The square wave $sq(t)$ *is the odd function of period* 2π *such that* $sq(t) = 1$ *for* $0 < t < \pi$ *and* $sq(\pi) = 0$*. In class we calculated its Fourier series.*

(b) $sq(t)$ *has minimal period* 2π*, but it is also a function of period* 4π*. Use the integral formulas for the Fourier coefficients to calculate its Fourier series, regarded as a function of period* 4π*. Comment on the relationship between your answer and the Fourier series for* $sq(t)$*.*

Use the Fourier series for $sq(t)$*, along with calculus and algebraic manipulations, to compute the Fourier series of each of the following functions without evaluating any of the integrals for the Fourier coefficients. In each case, sketch a graph of the function, as well, and give the minimal period.*

(c) $sq(t - \frac{\pi}{4})$.

(d) $1 + 2\,sq(2\pi t)$.

(e) *The* $f(t)$ *of* `[B]` *in the* `Fourier Coefficients` *Applet explored in this session.*

(f) *The periodic function* $g(t)$ *with period* 2π *such that* $g(t) = t$ *for* $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$ *and* $g(t) = \pi - t$ *for* $\frac{\pi}{2} \le t \le \frac{3\pi}{2}$*. (Hint: what is* $g'(t)$ *in terms of* $sq(t)$*?)*

Solution: **(a)** The angle difference formula for sine gives

$$2\sin(t - \frac{\pi}{3}) = 2\left(\cos(\frac{\pi}{3})\sin t - \sin(\frac{\pi}{3})\cos t\right) = -\sqrt{3}\cos t + \sin t$$

and this is the Fourier series. (If you don't remember the angle difference formula, you can use the complex exponential!: $\sin(t - \frac{\pi}{3}) = \text{Im}(e^{i(t-\pi/3)}) = \text{Im}(e^{-i\pi/3}e^{it}) = \text{Im}((\frac{1}{2} - \frac{\sqrt{3}}{2}i)(\cos t + i\sin t)) = -\frac{\sqrt{3}}{2}\cos t + \frac{1}{2}\sin t$.)

(b) $sq(t)$ is still odd, so $a_n = 0$, and, with $L = 2\pi$,

$$\begin{aligned}
b_n &= \frac{2}{2\pi}\int_0^{2\pi} sq(t)\sin(nt/2)\,dt \\
&= \frac{1}{\pi}\left(\int_0^{\pi} \sin(nt/2)\,dt + \int_{\pi}^{2\pi} -\sin(nt/2)\,dt\right) \\
&= -\frac{2}{n\pi}\cos(nt/2)\Big|_0^{\pi} + \frac{2}{n\pi}\cos(nt/2)\Big|_{\pi}^{2\pi} \\
&= \frac{2}{\pi n}(-\cos(\pi n/2) + 1 + \cos(2\pi n/2) - \cos(\pi n/2)) = \frac{2}{\pi n}c_n,
\end{aligned}$$

where $c_n = 1 - 2\cos(\frac{\pi n}{2}) + \cos(\pi n)$. We evaluate c_n for some small values of n:

n	$\cos(\frac{\pi n}{2})$	$\cos(\pi n)$	c_n
0	1	1	0
1	0	-1	0
2	-1	1	4
3	0	-1	0

and then things repeat. So $b_n = 0$ unless $n = 2, 6, 10, \ldots$, and for such n, $b_n = \frac{8}{\pi n}$. The Fourier series is $sq(t) = \frac{8}{\pi}(\frac{1}{2}\sin(\frac{2t}{2}) + \frac{1}{6}\sin(\frac{6t}{2}) + \cdots)$ *This is the same series as the Fourier series for $sq(t)$ when it is regarded as having period 2π.* The numbering of the terms is different—only every fourth term is nonzero instead of every other term—but the series itself is identical.

(c) We know $sq(t - \frac{\pi}{4}) = \frac{4}{\pi}(\sin(t - \frac{\pi}{4}) + \frac{1}{3}\sin(3t - \frac{3\pi}{4})) + \cdots)$.
Now $\sin(nt - \frac{n\pi}{4}) = -\sin(\frac{n\pi}{4})\cos(nt) + \cos(\frac{n\pi}{4})\sin(nt)$ and

$n =$	1	3	5	7
$-\sin(n\pi/4)$	$-\sqrt{2}/2$	$-\sqrt{2}/2$	$\sqrt{2}/2$	$\sqrt{2}/2$
$\cos(n\pi/4)$	$\sqrt{2}/2$	$-\sqrt{2}/2$	$-\sqrt{2}/2$	$\sqrt{2}/2$

so,

$$sq\left(t - \frac{\pi}{4}\right) = \frac{2\sqrt{2}}{\pi}\left(-\cos(t) - \frac{1}{3}\cos(3t) + \frac{1}{5}\cos(5t) + \frac{1}{7}\cos(7t) - - + + \cdots \right.$$
$$\left. + \sin(t) - \frac{1}{3}\sin(3t) - \frac{1}{5}\sin(5t) + \frac{1}{7}\sin(7t) + - - + \cdots \right).$$

(d) $1 + 2\,sq(2\pi t) = 1 + \frac{8}{\pi}(\sin(2\pi t) + \frac{1}{3}\sin(6\pi t) + \frac{1}{5}\sin(10\pi t) + \cdots)$.

(e) $f(t) = \frac{\pi}{4}\,sq(t + \frac{\pi}{2}) = \sin(t + \frac{\pi}{2}) + \frac{1}{3}\sin(3(t + \frac{\pi}{2})) + \cdots$.
Now $\sin(\theta + \frac{\pi}{2}) = \cos\theta$ and $\sin(\theta + \frac{3\pi}{2}) = -\cos\theta$, so

$$f(t) = \cos t - \frac{1}{3}\cos(3t) + \frac{1}{5}\cos(5t) + \cdots.$$

(f) $g(t)$ is odd so it's given by a sine series. $g'(t) = \frac{4}{\pi}sq(t)$, so the Fourier series of $g(t)$ is the integral of the Fourier series of $\frac{4}{\pi}sq(t)$:

$$g(t) = \frac{4}{\pi}\left(\sin(t) - \frac{1}{3^2}\sin(3t) + \frac{1}{5^2}\sin(5t) - \cdots\right).$$