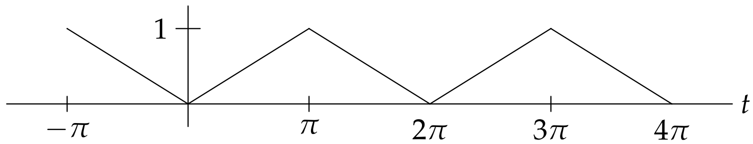
Example: Damped Harmonic Oscillator

Example. Let $f(t)$ be the triangle wave shown in figure 1. Solve the differential equation

$$\ddot{x} + 2\dot{x} + 9x = f(t).$$



Solution. Using a previous example, or computing directly, we have the Fourier series for $f(t)$ is

$$f(t) = \frac{1}{2} - \frac{4}{\pi^2}\left(\cos t + \frac{\cos 3t}{3^2} + \frac{\cos 5t}{5^2} + \ldots\right).$$

We follow the same steps as in the example in the previous note.

Step 1: Solving for the individual components:

Solve:

$$\ddot{x}_n + 2\dot{x}_n + 9x_n = \cos nt \tag{1}$$

If $n = 0$ we get $x_{n,p} = \frac{1}{9}$.

For $n \geq 1$ we have

Complex replacement: $\ddot{z}_n + 2\dot{z}_n + 9z_n = e^{int}, \quad x_n = \mathrm{Re}(z_n)$

Exponential Response formula: $z_{n,p} = \dfrac{e^{int}}{9 - n^2 + 2in}$.

Polar coords: $9 - n^2 + 2in = R_n e^{i\phi_n}$, where

$R_n = \sqrt{(9 - n^2)^2 + 4n^2}$ and $\phi_n = \mathrm{Arg}(9 - n^2 + 2in) = \tan^{-1}\dfrac{2n}{9 - n^2}$

(since the complex number is in the first or second we must take the arctangent between 0 and π).

Thus, $z_{n,p} = \dfrac{1}{R_n} e^{i(nt - \phi_n)}$, which implies $x_{n,p} = \dfrac{1}{R_n}\cos(nt - \phi_n)$

Step 2: Superposition. To make things easier in step one we did not include the Fourier coefficients of the input in the DE (1). To use superposition we need to include them here.

$$x_{\mathrm{sp}}(t) = \frac{1}{18} - \frac{4}{\pi^2}\left(\frac{\cos(t - \phi_1)}{R_1} + \frac{\cos(3t - \phi_3)}{3^2 R_3} + \frac{\cos(5t - \phi_5)}{5^2 R_5} + \ldots\right),$$

with the formulas for R_n and ϕ_n as above.