18.03SC Practice Problems 23

Fourier Series: Harmonic response

Solution suggestions

1. *Let $f(t)$ denote the even function $f(t)$ which is periodic of period 2π and such that $f(t) = |t|$ for $-\pi < t < \pi$. Graph $f(t)$.*

Here is the graph of $f(t)$.

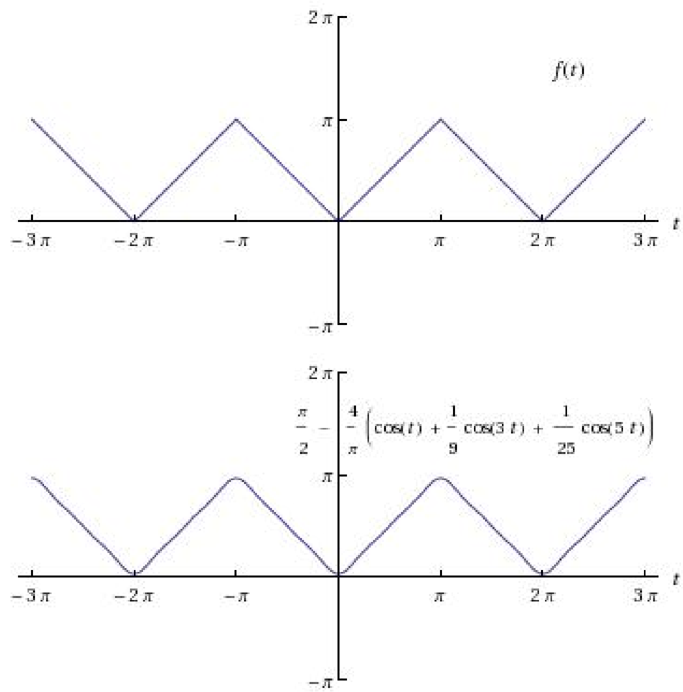

Figure 1: Graph of $f(t)$ over three periods. Below it is the graph of the first four nonzero terms of the Fourier series for $f(t)$. You can make all these graphs in Mathematica with the `Plot` and `Piecewise` packages.

In lecture we found that the Fourier series of $f(t)$ is given by

$$f(t) = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos(t) + \frac{\cos(3t)}{3^2} + \frac{\cos(5t)}{5^2} + \cdots\right)$$

Now we want to alter $f(t)$ to produce a function $g(t)$ whose graph is the same as that of $f(t)$ but is compressed (or expanded) horizontally so that the angular frequency is ω. What is the formula for $g(t)$ in terms of $f(t)$? Use the Fourier series for $f(t)$ and a substitution to find the Fourier series for the function $g(t)$.

The base angular frequency of f was 1, so the new function should have formula $g(t) = f(\omega t)$.

From this formula we get that the Fourier series for g has the same coefficients as that for f, but the frequencies are multiplied by ω:

$$g(t) = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos(\omega t) + \frac{\cos(3\omega t)}{3^2} + \frac{\cos(5\omega t)}{5^2} + \cdots\right)$$

2. *Next drive a simple harmonic oscillator with the function $f(t)$ from* **(1)***. This gives the differential equation*

$$\ddot{x} + \omega_n^2 x = f(t).$$

Find a periodic solution, when one exists, as a Fourier series.

By a superposition argument, the driven oscillator experiences resonance (and so does not have a periodic solution) if and only if the natural frequency ω_n of the system coincides with the angular frequency of a term with nonzero coefficient in the Fourier expansion of the driving function f.

So, in our case, we get periodic solutions for all $\omega_n \neq 0, 1, 3, 5, 7, \ldots$. Under this condition, we can use the first formula in the second box in the introduction to this problem session to read off a response to each Fourier series component: $\ddot{x} + \omega_n^2 x = \frac{\cos(k\omega t)}{k^2}$ has periodic solution $x(t) = \frac{\cos(k\omega t)}{k^2(\omega_n^2 - k^2)}$.

This means that the original equation has the following periodic solution:

$$x_p(t) = \frac{\pi}{2\omega_n^2} - \frac{4}{\pi}\left(\frac{\cos t}{\omega_n^2 - 1} + \frac{\cos 3t}{9(\omega_n^2 - 9)} + \frac{\cos 5t}{25(\omega_n^2 - 25)} + \ldots\right),$$

provided $\omega_n \neq 0, 1, 3, 5, 7, \ldots$.

3. *Now drive the same harmonic oscillator with the function $g(t)$ from* **(1)** *of angular frequency ω, obtaining the following differential equation for the response:*

$$\ddot{x} + \omega_n^2 x = g(t).$$

Again, find a periodic solution, when one exists.

Again we use superposition to combine termwise solutions, but this time, each term has angular frequency $k\omega$. We replace k by $k\omega$ everywhere, obtaining that the new equation has periodic solution

$$x_p = \frac{\pi}{2\omega_n^2} - \frac{4}{\pi}\left(\frac{\cos \omega t}{\omega_n^2 - \omega^2} + \frac{\cos(3\omega t)}{9(\omega_n^2 - 9\omega^2)} + \frac{\cos(5\omega t)}{25(\omega_n^2 - 25\omega^2)} + \ldots\right),$$

provided $\omega_n \neq 0, omega, 3\omega, 5\omega, ...$

4. *Suppose that ω is fixed, but that we can vary ω_n. This would be the case, for example, if we had a radio receiver and wanted to pick up (amplify) radio signals at or near a certain angular frequency. Then we would set the capacitance so that the natural frequency of the circuit would be some (variable) ω_n.*

You might have already answered this question in your solutions to **(2)** *and* **(3)***, but at what values of ω_n does the harmonic oscillator fail to have a periodic system response? Describe the system response when ω_n is just larger or just smaller than one of those values.*

As mentioned above, the harmonic oscillator fails to have a periodic system response when it is in resonance with the input - i.e., when ω_n is an odd multiple of ω or 0.

When ω_n is just larger or just smaller than one of the angular frequences of the driving function, the gain for that frequency component is large, so the Fourier series for the solution would have a large coefficient for that frequency, effectively picking out that component. This would be the perhaps simplified version of the effect we hope to capitalize on when tuning a radio to a particular frequency.

5. *Are there frequencies at which there is more than one periodic solution?*

The answer is yes, for special frequencies. The homogeneous solution for the system is

$$x_h = c_1 \cos(\omega_n t) + c_2 \sin(\omega_n t).$$

If the steady periodic solution, x_p, found in problem 4 has a period in common with x_h then we can add them together to get many solutions with that common period.

The base period of x_h is $2\pi/\omega_n$. From problem 4 we know that x_p is periodic with base period $2\pi/\omega$ provided $\omega_n \neq 0, \omega, 3\omega, 5\omega, \ldots$. (For these values of ω_n the solution x_p is not periodic.)

The functions x_h and x_p have a common period when some positive integer multiple of the base period of x_h equals a multiple of the base period of x_p. That is, when

$$M \cdot \frac{2\pi}{\omega_n} = N \cdot \frac{2\pi}{\omega}$$

for some positive integers M and N. This implies

$$\omega_n = \frac{M}{N} \cdot \omega.$$

We have our answer: There is more than one periodic solution if ω_n is any positive rational multiple of ω except $1, 3, 5, \ldots$