18.03SC Practice Problems 23

Fourier Series: Harmonic response

If $g(x)$ is a piecewise continuous periodic function and $2L$ is a period, then

$$g(x) = \frac{a_0}{2} + a_1 \cos\left(\frac{\pi x}{L}\right) + a_2 \cos\left(\frac{2\pi x}{L}\right) + \cdots + b_1 \sin\left(\frac{\pi x}{L}\right) + b_2 \sin\left(\frac{2\pi x}{L}\right) + \cdots$$

The Fourier coefficients are defined as the numbers fitting into this expression. They can be calculated using the integral formulas

$$a_n = \frac{1}{L}\int_{-L}^{L} g(x)\cos\left(\frac{n\pi x}{L}\right)\,dx\,, \qquad b_n = \frac{1}{L}\int_{-L}^{L} g(x)\sin\left(\frac{n\pi x}{L}\right)\,dx$$

$\ddot{x} + \omega_n^2 x = A\cos(\omega t)$ has solution $A\dfrac{\cos(\omega t)}{\omega_n^2 - \omega^2}$ and
$\ddot{x} + \omega_n^2 x = A\sin(\omega t)$ has solution $A\dfrac{\sin(\omega t)}{\omega_n^2 - \omega^2}$ as long as $\omega \neq \omega_n$.

1. Let $f(t)$ denote the even function $f(t)$ which is periodic of period 2π and such that $f(t) = |t|$ for $-\pi < t < \pi$. Graph $f(t)$.

In lecture we found that the Fourier series of $f(t)$ is given by

$$f(t) = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos(t) + \frac{\cos(3t)}{3^2} + \frac{\cos(5t)}{5^2} + \cdots\right)$$

Now we want to alter $f(t)$ to produce a function $g(t)$ whose graph is the same as that of $f(t)$ but is compressed (or expanded) horizontally so that the angular frequency is ω. What is the formula for $g(t)$ in terms of $f(t)$? Use the Fourier series for $f(t)$ and a substitution to find the Fourier series for the function $g(t)$.

2. Next drive a simple harmonic oscillator with the function $f(t)$ from **(1)**. This gives the differential equation

$$\ddot{x} + \omega_n^2 x = f(t).$$

Find a periodic solution, when one exists, as a Fourier series.

3. Now drive the same harmonic oscillator with the function $g(t)$ from **(1)** of angular frequency ω, obtaining the following differential equation for the response:

$$\ddot{x} + \omega_n^2 x = g(t).$$

Again, find a periodic solution, when one exists.

4. Suppose that ω is fixed, but that we can vary ω_n. This would be the case, for example, if we had a radio receiver and wanted to pick up (amplify) radio signals at or near a certain angular frequency. Then we would set the capacitance so that the natural frequency of the circuit would be some (variable) ω_n.

You might have already answered this question in your solutions to **(2)** and **(3)**, but at what values of ω_n does the harmonic oscillator fail to have a periodic system response? Describe the system response when ω_n is just larger or just smaller than one of those values.

5. Are there frequencies at which there is more than one periodic solution?

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