Part I Problems and Solutions

Problem 1: *For each spring-mass system, find whether pure resonance occurs, without actually calculating the solution.*

a) $2x'' + 10x = F(t)$; $F(t) = 1$ *on (0,1),* $F(t)$ *is odd, and of period 2;*

b) $x'' + 4\pi^2 x = F(t)$; $F(t) = 2t$ *on (0,1),* $F(t)$ *is odd, and of period 2;*

c) $x'' + 9x = F(t)$; $F(t) = 1$ *on* $(0, \pi)$, $F(t)$ *is odd, and of period* 2π

Solution: Consider

$$mx'' + kx = F(t)$$

The natural frequency of this spring-mass system is

$$\omega_0 = \sqrt{\frac{k}{m}}$$

The typical term of the Fourier expansion of $F(t)$ is $\cos \frac{n\pi}{L} t, \sin \frac{n\pi}{L} t$; thus we get pure resonance if and only if the Fourier series has a term of the form $\cos \frac{n\pi}{L} t$ or $\sin \frac{n\pi}{L} t$, where $\frac{n\pi}{L} = \omega_0$.

a) $\omega_0 = \sqrt{5}$ for spring-mass system, and $L = 1$. Fourier series is $\sum b_n \sin n\pi t$; $n\pi \neq \sqrt{5}$, so *no resonance*.

b) $\omega_0 = 2\pi$, $L = 1$. Fourier series is $\sum b_n \sin n\pi t$, and $n\pi = 2\pi$ if $n = 2$. Thus, *do get resonance*.

c) $\omega_0 = 3$. Fourier series is a sine series ($F(t)$ is odd): $F(t) = \sum b_n \sin nt$ — all odd n occur, so $n = 3$ occurs and *do get resonance*.

Problem 2: *Find a periodic solution as a Fourier series to* $x'' + 3x = F(t)$, *where* $F(t) = 2t$ *on* $(0, \pi)$, $F(t)$ *is odd, and has period* 2π.

Solution: Input: $F(t) = 4(\sin t - \frac{1}{2}\sin 2t + \frac{1}{3}\sin 3t - \ldots) = 4\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sin nt}{n}$

Solve in pieces: $x_n'' + 3x_n = \sin nt \Rightarrow x_n = \dfrac{\sin nt}{3 - n^2}$

Use superposition: (remember coefficients from the input)

$$x = 4\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sin nt}{n(3-n^2)} = 4\left(\frac{\sin t}{2} + \frac{\sin 2t}{2} - \frac{\sin 3t}{18} + \frac{\sin 4t}{52} - \ldots\right)$$