IVP's: Longer Examples

The fish population in a lake is not reproducing fast enough and the population is decaying exponentially with decay rate k. A program is started to stock the lake with fish. Three different scenarios are discussed below.

Example 1. A program is started to stock the lake with fish at a constant rate of r units of fish/year. Unfortunately, after 1/2 year the funding is cut and the program ends. Model this situation and solve the resulting DE for the fish population as a function of time.

Solution. Let $x(t)$ be the fish population and let $A = x(0^-)$ be the initial population. Exponential decay means the population is modeled by

$$\dot{x} + kx = f(t), \quad x(0^-) = A \tag{1}$$

where $f(t)$ is the rate fish are being added to the lake. In this case

$$f(t) = \begin{cases} r & \text{for } 0 < t < 1/2 \\ 0 & \text{for } 1/2 < t. \end{cases}$$

First, write f in *'u-format'*: $f(t) = r(1 - u(t - 1/2))$.
Next, take the Laplace transform and solve for $X(s)$.
$F(s) = \mathcal{L}(f)(s) = \frac{r}{s} - \frac{r}{s}e^{-s/2}$.
$\Rightarrow \quad sX - x(0^-) + kX = F(s) \quad \Rightarrow \quad (s+k)X - A = \frac{r}{s}(1 - e^{-s/2})$
$\Rightarrow \quad X(s) = \frac{A}{s+k} + \frac{r}{s(s+k)}(1 - e^{-s/2})$.
To find $x(t)$ we temporarily ignore the factor of $e^{-s/2}$ and take Laplace inverse of what's left. (using partial fractions).

$$\mathcal{L}^{-1}\left(\frac{A}{s+k}\right) = Ae^{-kt}, \qquad \mathcal{L}^{-1}\left(\frac{r}{s(s+k)}\right) = \frac{r}{k}(1 - e^{-kt}).$$

The t-translation formula says

$$\mathcal{L}^{-1}\left(\frac{re^{-s/2}}{s(s+k)}\right) = u(t - 1/2)\frac{r}{k}(1 - e^{-k(t-1/2)}).$$

Putting it all together we get (in u and cases format).

$$\begin{aligned} x(t) &= Ae^{-kt} + \frac{r}{k}(1 - e^{-kt}) - u(t - 1/2)\frac{r}{k}(1 - e^{-k(t-1/2)}) \\ &= \begin{cases} Ae^{-kt} + \frac{r}{k}(1 - e^{-kt}) & \text{for } 0 < t < 1/2 \\ Ae^{-kt} - \frac{r}{k}(e^{-kt} + e^{-k(t-1/2)}) & \text{for } 1/2 < t. \end{cases} \end{aligned}$$

Example 2. (Periodic on/off) The program is refunded and the have enough money to stock at a constant rate of r for the first half of each year. Find $x(t)$ in this case.

Solution. All that's changed from example 1 is the input function $f(t)$. We write it in cases-format and translate that to u-format so we can take the Laplace transform.

$$\begin{aligned} f(t) &= \begin{cases} r & \text{for } 0 < t < 1/2 \\ 0 & \text{for } 1/2 < t < 1 \\ r & \text{for } 0 < t < 3/2 \\ 0 & \text{for } 3/2 < t < 2 \\ & \ldots \end{cases} \\ &= r(1 - u(t - \frac{1}{2}) + u(t-1) - u(t - \frac{3}{2}) + \ldots) \end{aligned}$$

The computations from here are essentially the same as in the previous example.

$\mathcal{L}(f) = \frac{r}{s}(1 - e^{-s/2} + e^{-s} - e^{-3s/2} + \ldots)$

$\Rightarrow\ X = \frac{A}{s+k} + \frac{r}{s(s+k)}(1 - e^{-s/2} + e^{-s} - \ldots)$

$\Rightarrow\ x(t) = Ae^{-kt} + \frac{r}{k}\left[(1 - e^{-kt}) - u(t-1/2)(1 - e^{-k(t-1/2)}) + \ldots\right]$

$$\Rightarrow\ x(t) = \begin{cases} Ae^{-kt} + \frac{r}{k} - \frac{r}{k}e^{-kt} & \text{for } 0 < t < \frac{1}{2} \\ Ae^{-kt} - \frac{r}{k}(e^{-kt} - e^{-k(t-1/2)}) & \text{for } \frac{1}{2} < t < 1 \\ \ldots & \\ Ae^{-kt} + \frac{r}{k} - \frac{r}{k}(e^{-kt} - e^{-k(t-1/2)} + \ldots + e^{-k(t-n)}) & \text{for } n < t < n + \frac{1}{2} \\ Ae^{-kt} - \frac{r}{k}(e^{-kt} - e^{-k(t-1/2)} + \ldots - e^{-k(t-n-1/2)}) & \text{for } n + \frac{1}{2} < t < n+1 \\ \ldots & \end{cases}$$

Factoring out e^{-kt} gives:

$$x(t) = \begin{cases} Ae^{-kt} + \frac{r}{k} - \frac{r}{k}e^{-kt}(1 - e^{k/2} + e^k - e^{3k/2} + \ldots + e^{nk}) & \text{for } n < t < n + 1/2 \\ Ae^{-kt} - \frac{r}{k}e^{-kt}(1 - e^{k/2} + e^k - \ldots - e^{k(n+1/2)}) & \text{for } n + 1/2 < t < n + 1. \end{cases}$$

Note that the constant term r/k is only present during periods of stocking.

Example 3. (Impulse train) The answer to the previous example is a little hard to read. We know from experience that impulsive input usually leads to simpler output. In this scenario suppose that once a year $r/2$ units of fish are dumped all at once into the lake. Find $x(t)$ in this case.

Solution. Once again, all that's changed from example 1 is the input function $f(t)$. The IVP is still given by equation (1).

$$f(t) = \frac{r}{2}\left(\delta(t) + \delta(t-1) + \delta(t-2) + \delta(t-3) + \ldots\right).$$

This is called an *impulse train*. Its Laplace transform is easy to find.

$$F(s) = \frac{r}{2}\left(1 + e^{-s} + e^{-2s} + e^{-3s} + \ldots\right).$$

One nice thing about delta functions is that they don't introduce any new terms into the partial fractions part of the problem.

$$\begin{aligned} sX(s) - x(0^-) + kX(s) &= \frac{r}{2}\left(1 + e^{-s} + e^{-2s} + e^{-3s} + \ldots\right). \\ \Rightarrow X(s) &= \frac{A}{s+k} + \frac{r}{2(s+k)}\left(1 + e^{-s} + e^{-2s} + e^{-3s} + \ldots\right). \end{aligned}$$

Laplace inverse is easy:

$$\mathcal{L}^{-1}\left(\frac{1}{s+k}\right) = e^{-kt} \quad \Rightarrow \quad \mathcal{L}^{-1}\left(\frac{e^{-ns}}{s+k}\right) = u(t-n)e^{-k(t-n)}.$$

Thus,

$$x(t) = Ae^{-kt} + \frac{r}{2}e^{-kt} + \frac{r}{2}u(t-1)e^{-k(t-1)} + \frac{r}{2}u(t-2)e^{-k(t-2)} + \frac{r}{2}u(t-3)e^{-k(t-3)} + \ldots$$

Here are graphs of the solutions to examples 2 and 3 (with $A = 0, k = 1$, $r = 2$). Notice how they settle down to periodic behavior.

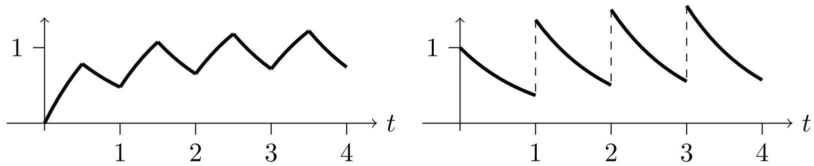

Fig. 1. Graphs from example 2 (left) and example 3 (right).