18.03SC Practice Problems 29

Solving IVP's

Rules for the Laplace transform

Definition:	$\mathcal{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}\,dt \quad \text{for} \quad \mathrm{Re}\,(s) \gg 0.$
Linearity:	$\mathcal{L}[af(t) + bg(t)] = aF(s) + bG(s).$
$\mathcal{L}^{-1}$:	$F(s)$ essentially determines $f(t)$ for $t > 0$.
s-shift rule:	$\mathcal{L}[e^{rt}f(t)] = F(s-r).$
s-derivative rule:	$\mathcal{L}[tf(t)] = -F'(s).$
t-derivative rule:	$\mathcal{L}[f'(t)] = sF(s) - f(0^-).$

Formulas for the Laplace transform

$$\mathcal{L}[1] = \frac{1}{s}, \quad \mathcal{L}[\delta(t-a)] = e^{-as}$$

$$\mathcal{L}[e^{rt}] = \frac{1}{s-r}, \quad \mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}[\cos(\omega t)] = \frac{s}{s^2+\omega^2}, \quad \mathcal{L}[\sin(\omega t)] = \frac{\omega}{s^2+\omega^2}$$

1. Let $f(t) = e^{-t}\cos(3t)$.

(a) From the rules and tables, what is $F(s) = \mathcal{L}[f(t)]$?

(b) Compute the derivative $f'(t)$ and its Laplace transform. Verify the t-derivative rule in this case.

2. Use the Laplace transform to find the unit step and impulse response of the operator $D + 2I$.

3. Use the Laplace transform to find the solution to $\dot{x} + 2x = t^2$ with initial condition $x(0) = 1$.

Solve each of the following by using the Laplace transform.

4. Find the unit impulse response of the operator $D + 3I$.

5. Find the solution to $\dot{x} + 3x = e^{-t}$ with rest initial conditions (so $x(0) = 0$).

6. Find the unit impulse response of the operator $D^3 + D$.

7. (a) Find the solution (for $t > 0$) to $\dot{x} + 3x = 1$ with $x(0) = 2$ by applying the Laplace transform to the equation.

(b) Find the solution (for $t > 0$) to $\dot{x} + 3x = 1 + 2\delta(t)$ with rest initial conditions by applying the Laplace transform to the equation.

(c) Explain the relationship between these two problems.