Part I Problems and Solutions

Solve the following IVP's by using the Laplace transform.

Problem 1: $y' - y = e^{3t},\ \ y(0^-) = 1$

Solution: We use the two formulas:

$$\mathcal{L}\left(y'\right) = -y(0^-) + s\mathcal{L}\left(y\right)$$

and

$$\mathcal{L}\left(y''\right) = -y'(0^-) - sy(0^-) + s^2\mathcal{L}\left(y\right)$$

$$\begin{aligned}
(s\mathcal{L}y - 1) - \mathcal{L}y &= \frac{1}{s-3} \\
(s-1)\mathcal{L}y &= 1 + \frac{1}{s-3} \\
\mathcal{L}y &= \frac{1}{s-1} + \frac{1}{(s-1)(s-3)} \\
&= \frac{1/2}{s-1} + \frac{1/2}{s-3} \\
y &= \frac{1}{2}e^t + \frac{1}{2}e^{3t}
\end{aligned}$$

Problem 2: $y'' - 3y' + 2y = 0, y(0^-) = 1, y'(0^-) = 1$

Solution:

$$\begin{aligned}
(s^2\mathcal{L}y - s - 1) - 3(s\mathcal{L}y - 1) + 2\mathcal{L}y = 0 \\
(s^2 - 3s + 2)\mathcal{L}y = s - 2 \\
\mathcal{L}y = \frac{1}{s-1} \\
y = e^t
\end{aligned}$$

Problem 3: $y'' + 4y = \sin t,\ \ y(0^-) = 1, y'(0^-) = 0$

Solution: $(s^2\mathcal{L}y - s) + 4\mathcal{L}y = \frac{1}{s^2+1}$, so $\mathcal{L}y = \frac{1}{(s^2+1)(s^2+4)} + \frac{s}{s^2+4}$.

(It's easier not to combine terms here).
Next, apply partial fractions to this expression for $\mathcal{L}y$, treating s^2 as a single variable u:

$$\frac{1}{(u+1)(u+4)} = \frac{1/3}{u+1} - \frac{1/3}{u+4}$$

Now put in $u = s^2$:

$$\mathcal{L}y = \frac{1/3}{s^2+1} - \frac{1/3}{s^2+4} + \frac{s}{s^2+4}$$

Thus,

$$y = \frac{1}{3}\sin t - \frac{1}{6}\sin 2t + \cos 2t$$

Problem 4: $y'' - 2y' + 2y = 2e^t, \;\; y(0^-) = 0, y'(0^-) = 1$

Solution:

$$\begin{aligned}
(s^2\mathcal{L}y - 1) - 2s\mathcal{L}y + 2\mathcal{L}y &= \frac{2}{s-1} \\
(s^2 - 2s + 2)\mathcal{L}y &= \frac{2}{s-1} + 1 = \frac{s+1}{s-1} \\
\mathcal{L}y &= \frac{s+1}{(s^2-2s+2)(s-1)} \\
&= \frac{2}{s-1} + \frac{3-2s}{s^2-2s+2} \\
&= \frac{2}{s-1} - \frac{2(s-1)}{(s-1)^2+1} + \frac{1}{(s-1)^2+1}
\end{aligned}$$

Note that we write the second term as an expression in $s-1$; the last term is what is left over. We then obtain our answer,

$$y = 2e^t - 2e^t\cos t + e^t\sin t$$

Problem 5: $y'' - 2y' + y = e^t, \;\; y(0^-) = 1, y'(0^-) = 0$

Solution:

$$\begin{aligned}
s^2\mathcal{L}y - s - 2(s\mathcal{L}y - 1) + \mathcal{L}y &= \frac{1}{s-1} \\
(s^2 - 2s + 1)\mathcal{L}y &= \frac{1}{s-1} + (s-2) \\
(s-1)^2\mathcal{L}y &= \frac{1}{s-1} + (s-1) - 1 \\
\mathcal{L}y &= \frac{1}{(s-1)^3} + \frac{1}{(s-1)} - \frac{1}{(s-1)^2}
\end{aligned}$$

$$y = \frac{t^2}{2}e^t + e^t - te^t$$

Problem 6: $x'' - 6x' + 8x = 2, \quad x(0^-) = x'(0^-) = 0$

Solution: Let $X(s) = \mathcal{L}(x(t))$. The transformed IVP is then $(s^2 - 6s + 8)X(s) = \frac{2}{s}$, since $x(0^-) = x'(0^-) = 0$. Thus,

$$X(s) = \frac{2}{s(s-2)(s-4)} = \frac{1}{4}\left(\frac{1}{s} - \frac{2}{s-2} + \frac{1}{s-4}\right)$$

Thus,

$$x(t) = \mathcal{L}^{-1}(X(s)) = \frac{1}{4}\left(1 - 2e^{2t} + e^{4t}\right)$$

Problem 7: *Solve the IVP* $x^{(4)} + 2x'' + x = e^{2t}; \; x(0^-) = x'(0^-) = x''(0^-) = x^{(3)}(0^-) = 0$

Solution: Let $X(s) = \mathcal{L}(x)$.

$$\begin{aligned}
(s^4 + 2s^2 + 1)X(s) &= \frac{1}{s-2} \\
X(s) &= \frac{1}{(s-2)(s^2+1)^2} \\
&= \frac{1}{25}\left(\frac{1}{s-2} - \frac{s+2}{s^2+1} - 5\frac{s+2}{(s^2+1)^2}\right) \\
x(t) = \mathcal{L}^{-1}(X) &= \frac{1}{50}\left(2e^{2t} - 2\cos t - 4\sin t - 5t\sin t - 10(\sin t - t\cos t)\right)
\end{aligned}$$

$$x(t) = \frac{1}{50}\left(2e^{2t} + (10t - 2)\cos t - (5t + 14)\sin t\right)$$

Problem 8: *Find the Laplace transform of $f(t) = (u(t) - u(t - 2\pi))\sin(t)$ by use of the t-shift rule.*

Solution: In this case we can write $\sin t = \sin(t - 2\pi)$ (since it is periodic with period 2π). Then using the shift rule
$\mathcal{L}\left(u(t-a)f(t-a)\right) = e^{-as}\mathcal{L}\left(f\right)$ we have

$$\mathcal{L}\left(u(t) - u(t - 2\pi)\sin(t - 2\pi)\right) = \frac{1}{s} - e^{-2\pi s}\mathcal{L}\left(\sin t\right) = \frac{1}{s} - e^{-2\pi s}\left(\frac{1}{s^2 + 1}\right)$$